DIFFEOMORPHISMS LYING IN ONE PARAMETER GROUPS AND EXTENSION OF STRATIFIED HOMEOMORPHISMS

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Abstract. In this note we first show using results of McDuff that for a C^r-manifold S, diffeomorphic to the interior of a compact manifold with boundary, the class of all C^r-diffeomorphisms lying in a one parameter group of S generates the connected component of 1_S in $Diff^r(S,S)$. Then we use this result to obtain two extension theorems for stratified maps defined on some strata of a stratified space $\mathcal{X}$. Our extension theorems hold for Mather's abstract stratified sets, for Whitney (b)-regular, Bekka (c)-regular, Verdier (w)-regular and Lipschitz-regular stratified spaces.

A.M.S. Classifications: Primary 58A35; 54C20; Secondary 57R52;

Key Words: One parameter groups. Regular stratifications.

1. Introduction. If S is a closed C^r-manifold, i.e. compact and without boundary, by denoting by $Diff^r(S,S)$ the group of all C^r-diffeomorphisms of S $(r = 0, \dots, \infty)$, it is well-known that the connected component $Diff^r_0(S,S)$ of the identity map $1_S : S \to S$ is a simple group (except possibly in the case $r = \dim S + 1$) **[3, 4, 10, 11, 12, 13, 20]**.

This theorem allows one to show that for a closed C^r-manifold, the image of the exponential map, i.e. the class E of all C^r-diffeomorphisms lying in a one parameter group of S, generates the whole of $Diff^r_0(S,S)$.

On the other hand, if S is a non-compact manifold the *"simplicity of $Diff^r_0(S,S)$"* fails to be true **[14]**, so the property that E generates the whole of $Diff^r_0(S,S)$ cannot be deduced and remains unproved. In 1984 J. Milnor **[15]** claimed, but without proof, that such a property ought to hold at least in the class of the C^∞-manifolds diffeomorphic to the interior of a compact manifold with boundary, (remark 1.7).

In this paper we first show, in §2, that Milnor's claim holds for all such C^r-manifolds and for every $r \geq 1$ (theorem 1). That is, if S is a manifold diffeomorphic to the interior $intM$ of a compact manifold with boundary M, then E generates the whole of $Diff^r_0(S,S)$. In other words, every element $f \in Diff^r_0(S,S)$ may be written as a composition of flows at time $t = 1$, $f = \phi^1_1 \circ \cdots \circ \phi^s_1$, of vector fields $\zeta^1, \dots, \zeta^s$ on S.

Then in §3 we apply theorem 1 in the context of regular stratifications to obtain two stratified extension theorems (theorems 2 and 3), first for abstract stratified sets **[8, 9]**, a category which contains (b)-regular **[21, 23]** and (c)-regular stratifications **[1, 2]**, and then for (w)-regular stratifications **[22]** (proposition 1) a class containing also Lipschitz stratifications **[18]**.

In theorem 2 we show that if $\mathcal{X} = (A, \Sigma)$ is an abstract stratified set and $S = A_k - A_{k-1}$ is the union of all k-strata of $\mathcal{X}$ every diffeomorphism $f \in Diff_0^1(S, S)$ (which extends continuously on A_{k-1} also) extends continuously on a neighborhood U of S in A. In fact, writing $f = \phi_1^1 \circ \cdots \circ \phi_1^s$, then thanks to the theorem of stratified lifting of vector fields **[8, 9]**, we can lift the vector fields ζ^i and hence the flows ϕ^i on a neighborhood U of S in A, and therefore we obtain by composition an extension of f which is a stratified homeomorphism defined on the neighborhood U of S in A (and which again extends continuously on A_{k-1}). Because such an extension is not on the whole of A we call theorem 2 of "weak extension".

The weak extension theorem may be improved in order to construct extensions which are stratified homeomorphisms, isotopic to the identity, of the whole stratified space $\mathcal{X}$ ("strong extension"). This occurs for example in theorem 2 when A_{k-1} is empty.

In a more general case this occurs if instead of a diffeomorphism $f : S \to S$ of $S = \cup_{\dim X = k} X$ we consider a homeomorphism $f = \cup_j f_{X_j}$, union of diffeomorphisms $f_{X_j} \in Diff_0^1(X_j, X_j)$, defined on a collection of strata $\{X_j\}_j$ of various dimensions provided that all X_j are compact. This is the content of theorem 3.

Such stratified extension theorems can be used **[17]** to show a transversality theorem (by isotopy) for substratified spaces of a stratified space, allowing us to develop geometric homology and cohomology theories in which the ambient space $\mathcal{X}$, its cycles and cocycles are abstract stratified sets or Bekka (c)-regular stratifications (cf. **[16]** chapter IV), analogous to the geometric theories of Goresky **[6]**.

We conclude the paper by Remark 5 in which we conjecture a possible interesting improvement of McDuff's results (theorem 1 and Corollary 2, **[14]**), using which one would obtain a global and stronger extension theorem for maps between stratified spaces.

I thank D. Trotman for many helpful discussions while writing this paper.

2. The image of the exponential map Exp^r generates $Diff_0^r(S, S)$.

Let $S = intM$ be a C^r-manifold diffeomorphic to the interior $intM$ of a compact C^r-manifold M with boundary.

Since $S = intM = M - \partial M$ with M compact, the simplicity of $Diff_0^r(S, S)$ fails to be true, but, in this case, all normal subgroups of $Diff_0^r(S, S)$ are completely classified by D. McDuff **[14]** and are in bijective correspondence with the lattice of subsets of $\{1, \ldots, k\}$ where k is the number of the connected components $N_1, \ldots, N_k$ of the boundary ∂M of M.

Thanks to the results of McDuff (Theorem 1 and Corollary 1.3), we will show that for such a manifold S, the diffeomorphisms which are images of the exponential map generate $Diff_0^r(S, S)$.

Let us fix $r \geq 1$, $r \neq \dim S + 1$, and denote by Exp^r the exponential map of S

$$Exp^r \;:\; \Gamma_0^r(S) \to Diff_0^r(S, S) \qquad , \qquad Exp^r(\zeta) = \phi_1$$

defined on the set $\Gamma_0^r(S)$ of all C^r-vector fields ζ of S admitting a global flow $\phi : S \times \mathbb{R} \to S$ (denoted also by $\phi = \{\phi_t : S \to S\}_{t \in \mathbb{R}}$) which associates to each ζ the diffeomorphism $\phi_1 : S \to S$ at time $t = 1$.

Obviously every image $\phi_1 = Exp^r(\zeta)$ is in $Diff_0^r(S, S)$, i.e. lies in the connected component of 1_S and moreover for every (other) $t \in \mathbb{R}$, $\phi_t \in Diff_0^r(S, S)$, since $\phi_t = \psi_1 = Exp^r(\eta)$ is the exponential of the vector field $\eta = t \cdot \zeta$.

On the other hand, denoting by $Exp^r(S) =< Exp^r\big(\Gamma_0^r(S)\big) >$ the subgroup of $Diff_0^r(S,S)$ generated by the image of the exponential map, with the same (well-known) proof as in the compact case, we immediately have that :

REMARK 1. *$Exp^r(S)$ is a normal subgroup of $Diff_0^r(S,S)$.* □

THEOREM 1. *If $S = intM$ is a C^r-manifold (diffeomorphic to the) interior of a compact C^r-manifold M with boundary, then :*

$$Exp^r(S) = Diff_0^r(S,S).$$

Proof. Let us write $S = M - \partial M$ with M a compact manifold with boundary. Then, the boundary ∂M of M is compact too and has a finite number of connected components, namely $N_1,\dots,N_k$, i.e. : $\partial M = \sqcup_{j=1}^k N_j$ (where $\sqcup$ means "disjoint union").

Using the same notations as in **[14]**, we let $K = \{1,\dots,k\}$, and for every $j \in K$,

$$G_j = \Big\{ g \in Diff_0^r(S,S) \mid g = id \text{ in a neighborhood of } N_j \Big\}$$

and for every subset $J \subseteq K$ (possibly empty) write

$$G_J = \Big\{ g \in Diff_0^r(S,S) \;\Big|\; g = id \;\text{ in a neighborhood of } \bigcup_{j\in J} N_j \Big\} = \bigcap_{j\in J} G_j.$$

It is immediate to verify that G_J is a normal subgroup of $Diff_0^r(S,S)$ and that for every $I, J \subseteq K$ we have moreover : " $I \subseteq J \Longleftrightarrow G_J \subseteq G_I$".

Therefore we obtain a lattice of normal subgroups $\big(\{G_J\}_{J\subseteq K}, \supseteq\big)$, corresponding to the lattice $\big(\{J\}_{J\subseteq K}, \subseteq\big)$ of subsets of K, and admitting the subgroup G_K of all diffeomorphisms with compact support in S as minimum element and the subgroup $G_\emptyset = Diff_0^r(S,S)$ as maximum element.

McDuff shows (Theorem 1) that *"E is a normal subgroup of $Diff_0^r(S,S)$ if and only if there exists a unique subset $J \subseteq K$ such that $G_J \supseteq E \supseteq [G_J, G_J]$* (where [,] denotes the subgroup of the commutators)".

By considering $Exp^r(S)$, which is a normal subgroup (remark 1), we deduce that there exists a unique subset J_0 of K such that

$$G_{J_0} \supseteq Exp^r(S) \supseteq [G_{J_0}, G_{J_0}].$$

On the other hand it is not difficult to verify that $G_i \not\supseteq Exp^r(S)$ for every $i \in K$.

In fact, for each $i = 1,\dots,k$ let us consider a non-zero vector field $\zeta^i_{N_i}$ on N_i admitting a global flow ϕ^i and lift it along a collar $C_i \equiv N_i \times [0,1]$ (where N_i is identified to $N_i \times 0$).

Let us consider then the vector field ζ^i defined on S by the formula:

$$\zeta^i(y) = \begin{cases} g(s)\cdot \zeta^i_{N_i}(x) & \text{if } y = (x,s) \in C_i = N_i \times]0,1] \\ \\ 0 & \text{if } y \in S - N_i \end{cases}$$

where $g \in C^r([0,1])$ is a smooth decreasing map verifying $g(0) = 1$ and $g^{(n)}(1) = 0$ $\forall n = 0,\dots,r$ (i.e. g vanishes in $s=1$ together all its derivatives).

Obviously, every vector field ζ^i admits again a global flow Φ^i (given on C_i by $\Phi^i\big((x,s),t\big) = \phi^i(x, g(s)t)$) and hence the diffeomorphism at time $t=1$, $\Phi^i_1 = Exp^r(\zeta^i)$

lies by definition in $Exp^r(S)$. But on the other hand since $\zeta_{N_i} \neq 0$ then $\phi^i_{1|N_i} \neq 1_{N_i}$ and so $\Phi^i_1 \notin G_i$.

Finally since for every $i \in K$, $G_i \not\supseteq Exp^r(S)$, we deduce that the only possible G_{J_0} which can contain $Exp^r(S)$ corresponds to the subset $J_0 = \emptyset$ and hence $G_{J_0} = G_\emptyset = Diff^r_0(S,S)$.

We have then,

$$Diff^r_0(S,S) \supseteq Exp^r(S) \supseteq [Diff^r_0(S,S), Diff^r_0(S,S)]\,,$$

which concludes the proof since $Diff^r_0(S,S)$ is a perfect group ([**14**], Corollary 1.3). □

3. Extension of stratified homeomorphisms.

All manifolds and diffeomorphisms in this section are considered of class C^1, similarly smooth will mean C^1 and thus we will write simply $Diff_0$ and Exp respectively for $Diff^1_0$ and Exp^1.

Theorem 1 above may be used for a stratified space $\mathcal{X} = (A, \Sigma)$ in order to obtain some theorems of extension of stratified maps defined on some strata of $\mathcal{X}$.

Definition 1. We recall that a *stratification* of a topological space A is a locally finite partition Σ of A into C^1 connected manifolds (called the *strata* of A) which satisfy the *frontier condition* : if X and Y are strata such that X intersects the closure of Y, then X is contained in the closure of Y. We write then $X < Y$ and denoting by $\partial Y = \overline{Y} - Y$ we have $\overline{Y} = Y \sqcup \left(\sqcup_{X<Y} X\right)$ and $\partial Y = \sqcup_{X<Y} X$, [**8**] (recall that $\sqcup$ = disjoint union).

Under such hypotheses the pair $\mathcal{X} = (A, \Sigma)$ is called a *stratified space* with *support* A and stratification Σ. The union of the strata of dimension $\leq k$ is called the *k-skeleton*, denoted by $\mathcal{X}_k$ or A_k.

Extra regularity conditions may then be imposed on the stratification Σ, such as to be an *abstract stratified set* in the sense of Mather [**8, 9**] when A is not necessarily embedded in a manifold, or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney [**21, 23**], or (c) of Bekka [**2**] or, when A is a subset of a C^2 manifold, to satisfy conditions (w) of Kuo-Verdier [**7, 22**], or (L) of Mostowski [**18**].

The stratified extension theorems of this section will be done first for $\mathcal{X}$ an abstract stratified set, then we show that the proofs work again for all types of regular stratified spaces listed above.

Definition 2. Let $\mathcal{X} = (A, \Sigma)$ be a stratified space.

A family $\mathcal{F} = \{(\pi_X, \rho_X, T_X)\}_{X\in\Sigma}$ is called *a system of control data* for $\mathcal{X}$ if for each stratum X of $\mathcal{X}$ we have that:

1) T_X is a neighbourhood of X in A (called *tubular neighbourhood of* X);
2) $\pi_X : T_X \to X$ is a continuous retraction of T_X onto X (called *projection on* X);
3) $\rho_X : T_X \to [0, \infty[$ is a continuous function such that $X = \rho_X^{-1}(0)$ (called the *distance function from* X)

and, furthermore, for every pair of adjacent strata $X < Y$, by considering the restriction maps $\pi_{XY} = \pi_{X|T_{XY}}$ and $\rho_{XY} = \rho_{X|T_{XY}}$, to the subset $T_{XY} = T_X \cap Y$, we have that :

5) the map $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0, \infty[$ is a smooth submersion (it follows in particular that $\dim X < \dim Y$);
6) for every stratum Z of $\mathcal{X}$ such that $Z > Y > X$ and for every $z \in T_{YZ} \cap T_{XZ}$ the following *control conditions* are satisfied :

i) $\pi_{XY}\pi_{YZ}(z) = \pi_{XZ}(z)$ (called the π-*control condition*)
ii) $\rho_{XY}\pi_{YZ}(z) = \rho_{XZ}(z)$ (called the ρ-*control condition*).

In what follows we will pose $T_X(\epsilon) = \rho_X^{-1}([0,\epsilon[)\,, \forall\, \epsilon \geq 0$, and without loss of generality will assume $T_X = T_X(1)$ [**5, 8**].

If A is Hausdorff, locally compact and admits a countable basis for its topology, the pair $(\mathcal{X}, \mathcal{F})$ is called an *abstract stratified set*. Since one usually works with a unique system of control data of $\mathcal{X}$, in what follows we will omit $\mathcal{F}$.

If $\mathcal{X}$ is an abstract stratified set, then A is metrizable and the tubular neighbourhoods $\{T_X\}_{X\in\Sigma}$ may (and will always) be chosen such that: "$T_{XY} \neq \emptyset$ if and only if $X < Y$ or $X > Y$ or $X = Y$" (see [**8**] page 41 and following).

Notice also that the notion of system of control data of $\mathcal{X}$, introduced by Mather in [**8**], is a fundamental tool allowing one to obtain a good (i.e. *controlled* cf. [**5, 8**]) stratified lifting ζ_{T_X} on a tubular neighbourhood T_X of every vector field ζ_X given on a stratum X. Such a lifting ζ_{T_X} admits a global flow $\phi_{T_X} : T_X \times \mathbb{R} \to T_X$ (when ζ_X admits it) which is furthermore a continuous map.

Definition 3. A *stratified map* $f : \mathcal{Y} \to \mathcal{X}$ between two stratified spaces $\mathcal{Y} = (B, \Sigma_{\mathcal{Y}})$ and $\mathcal{X} = (A, \Sigma)$ is a continuous map $f : B \to A$ which sends each stratum R of $\mathcal{Y}$ into a unique stratum S of $\mathcal{X}$, such that the restriction $f_R : R \to S$ is smooth. We call such a map f a *stratified homeomorphism* if f is a global homeomorphism and each f_R is a diffeomorphism.

Theorem 2 (of weak stratified extension). *Let $\mathcal{X} = (A, \Sigma)$ be a compact abstract stratified set, A_k its k-skeleton and $S = A_k - A_{k-1}$ the union of all its k-strata.*

Every homeomorphism $f : A_k \to A_k$, whose restriction f_S lies in $Diff_0(S,S)$, may be extended to a stratified homeomorphism $\tilde{f} : U \cup \partial S \to V \cup \partial S$ where U and V are two neighbourhoods of S in A.

Proof. Since A is compact then so is every closed subset, and in particular the union A_k of all its strata of dimension $\leq k$ [**8, 9**].

Let us write $\{X_j\}_j$ for the family of the strata of dimension k of $\mathcal{X}$. Then $S = A_{k+1} - A_k = \sqcup_j X_j$ and $\partial S = \overline{S} - S = \cup_j \partial X_j$.

Since by the frontier condition, every stratum X of $\mathcal{X}$ verifies $\overline{X} = X \sqcup \big(\sqcup_{R<X} R\big)$ $(\sqcup_{R<X} R = \partial X)$, then for every X_j one and only one of two following cases occurs :

i) there exist no strata $R < X_j$, hence $\overline{X_j} - X_j = \emptyset$ and X_j is a compact manifold (X is then called of *maximal depth*).

ii) there exists a stratum $R < X_j$, hence $\overline{X_j} - X_j \neq \emptyset$, X_j is non compact, but it is diffeomorphic to the interior of a compact manifold with boundary [**19**].

By the known results if X_j is compact (see introduction), and by theorem 1 if X_j is non compact, we have in both cases that $Diff_0(X_j, X_j) = Exp(X_i)\ \forall\ j$. On the other hand because $S = \sqcup_j X_j$ is a disjoint union and each diffeomorphism $f \in Diff_0(S,S)$, being isotopic to the identity, must fix the connected components S (i.e. $f(X_j) = X_j$), then we also have $Diff_0(S,S) = Exp(S)$.

Let us fix now a map $f : A_k \to A_k$ as in the statement of the theorem.

Because $Diff_0(S,S) = Exp(S)$ and since the restriction f_S lies in $Diff_0(S,S)$, we can rewrite f_S by

$$f_S = \phi_1^1 \circ \cdots \circ \phi_1^s \quad \text{with} \quad \phi_1^i = Exp(\zeta^i)\ ,$$

where for all $i = 1, \ldots, s$, ζ^i is a vector field on S admitting a global flow $\phi^i : S \times \mathbb{R} \to S$.

Now we want to extend $f_S : S \to S$ (and f) by lifting the vector fields ζ^i of the flows ϕ_t^i, and this will be possible by considering a controlled lifting of every ζ^i [**5, 8**].

For every $i = 1, \dots s$ and for every connected component X of S, i.e. for every k-stratum (above noted by X_j) of A, let us consider the restriction ζ^i_X of ζ^i on X and $\zeta^i_{T_X} = \{\zeta^i_{T_{XY}}\}_{Y \geq X}$ (with $T_{XY} = T_X \cap Y$) a stratified controlled lifting of ζ^i_X on a stratified tubular neighborhood $T_X = \cup_{Y \geq X} T_{XY}$ of a fixed system of control data $\mathcal{F} = \{(\pi_Z, \rho_Z, T_Z)\}_{Z \in \Sigma}$ of $\mathcal{X}$ [**8**].

Because every vector field ζ^i_X has a global flow ϕ^i_X (restriction of ϕ^i), then thanks to the hypotheses of control, the same property holds for the lifting $\zeta^i_{T_X}$ and hence its flow $\Phi^i_{T_X} : T_X \times \mathbb{R} \to T_X$ is a stratified flow extending continuously ϕ^i_X [**5, 8**].

By considering then the tubular neighborhood $T_S = \cup_{\dim X = k} T_X$ of S in A and, for every $i = 1, \dots, r$ the stratified one parameter group $\phi^i_{T_S} = \cup_{\dim X = k} \phi^i_{T_X} : T_S \times \mathbb{R} \to T_S$ we obtain that the map composition of the "sections at time $t = 1$" of the $\phi^i_{T_S}$, i.e. :

$$h \;=\; [\phi^1_{T_S}]_1 \circ \dots \circ [\phi^s_{T_S}]_1 \quad : \quad T_S \longrightarrow T_S \;,$$

is a stratified homeomorphism of the tubular neighborhood T_S, a diffeomorphism restricted to each stratum of T_S, and extends $f : S \to S$.

The proof follows, by applying to h and h^{-1} the lemma below. $\square$

Lemma. *Let $T_S \subseteq A$ be a stratified tubular neighborhood of S in $\mathcal{X} = (A, \Sigma)$ and let $h : T_S \to T_S$ be a continuous map whose restriction $h_S : S \to S$ to S extends continuously on $\partial S = \overline{S} - S$ by a continuous map $h_{\partial S} : \partial S \to \partial S$.*

There exists a neighborhood U of S in A (contained in T_S) such that the map $h_U \cup h_{\partial S} : U \cup \partial S \to T_S \cup \partial S$ is continuous.

Proof. The support A of the abstract stratified set $\mathcal{X}$ is metrizable so we can fix a metric $d(\ ,\ )$ on A. Denote moreover by $\mathcal{F}$ the system of control data of $\mathcal{X}$ and by $\{\pi_X : T_X \to X\}_{X \in \Sigma}$ its family of projections.

Because $S = A_k - A_{k-1} = \sqcup_{\dim X = k} X$ then the stratified tubular neighborhood T_S of S is a disjoint union $T_S = \sqcup_{\dim X = k} T_X$ where each T_X is the stratified tubular neighborhood the k-stratum X and where the elements of the family $\{T_X\}_{\dim X = k}$ are pairwise disjoint. We can consider then a global projection map $\pi_S = \cup_{\dim X = k} \pi_X : T_S \to S$ of T_S on S and we will denote, for every $y \in T_S$, $y' = \pi_S(y)$.

Let us consider the following subset of T_S :

$$U = \Big\{ y \in T_S \quad \Big| \quad d(y, y') < \frac{1}{2} d(y', \partial S) \quad \text{and} \quad d(h(y), h(y')) < d(y', \partial S) \Big\} \,.$$

Since $\partial S = \overline{S} - S$ is disjoint from S and T_S, for every $y \in T_S$ we have $d(y', \partial S) > 0$ and so U is clearly not empty and, by continuity of h, is an open neighborhood of S in A.

Since the restriction $h_S : S \to S$ extends continuously on ∂S by the continuous map $h_{\partial S}$, then for every fixed $x \in \partial S$ and $\epsilon > 0$ there exists a $\delta \in \,]0, \frac{\epsilon}{2}[$ such that $d(h(z), h(x)) < \frac{\epsilon}{2}$, $\forall\, z \in B(x, \delta) \cap S$.

If now $y \in U \cap B\big(x, \frac{\delta}{2}\big)$ by definition of U we immediately have

$$d(y, y') < \tfrac{1}{2} d(y', \partial S) < \tfrac{1}{2} d(y', x) \quad \text{and} \quad d(y', x) < d(y', y) + d(y, x) < \tfrac{1}{2} d(y', x) + d(y, x)$$

by which

$$d(y', x) < 2 d(y, x) < \delta \,, \qquad \text{thus} \quad y' \in B(x, \delta) \cap S \,, \qquad \text{and hence} \quad d(h(y'), h(x)) < \tfrac{\epsilon}{2} \,.$$

On the other hand, again by definition of U we also have

$$d((h(y), h(y')) < d(y', \partial S) < d(y', x) < \delta < \frac{\epsilon}{2}$$

which allows one to conclude that $\forall\, y \in B(x, \frac{\delta}{2})$ we have :

$$d(h(y), h(x)) < d((h(y), h(y')) + d(h(y'), h(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,. \qquad \square$$

THEOREM 3. *Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set. If $\{X_j\}_j$ is a family of compact strata of $\mathcal{X}$, then every homeomorphism $f : \sqcup_j X_j \to \sqcup_j X_j$, whose restriction f_{X_j} lies in $Diff_0(X_j, X_j)$, extends to a stratified homeomorphism $\tilde{f} : A \to A$, isotopic to the identity, on the whole of A.*

Proof. Let us write every restriction $f_{X_j} \in Diff_0(X_j, X_j)$ as

$$f_{X_j} = [\phi^1_{X_j}]_1 \circ \ldots \circ [\phi^{r_j}_{X_j}]_1 \qquad \text{where} \qquad [\phi^r_{X_j}]_1 = Exp(\zeta^r_{X_j}), \quad \forall\, r = 1, \ldots, r_j$$

and $\zeta^r_{X_j}$ is a smooth vector field on X_j (for all $r = 1, \ldots, r_j$).

Then by the properties of abstract stratified sets **[8, 9]** for every $\zeta^r_{X_j}$ there exists a stratified (controlled) vector field $\zeta^r_{T_{X_j}}$ a lifting of $\zeta^r_{X_j}$ on the tubular neighbourhood T_{X_j} of X_j. Such a lifted vector field admits (thanks to the control conditions) a stratified global flow $\phi^r_{T_{X_j}}$ on T_{X_j} **[8]** which is moreover continuous on T_{X_j} and extends $\phi^r_{X_j}$.

Let us modify the modulus (but only the modulus) of $\zeta^r_{T_{X_j}}$ in T_{X_j} by a smooth and decreasing function $g : [0, 1] \to [0, 1]$ such that $g(s) = 1$ if $s \leq \frac{1}{2}$ and $g(s) = 0$ if $s \geq 1$, in such a way to obtain the new vector field $\tilde{\zeta}^r_{T_{X_j}}(x) = g(\rho_{X_j}(x))\zeta^r_{T_{X_j}}(x)$ admitting again a global flow (namely $\tilde{\phi}^r_{T_{X_j}}$) and which moreover extends smoothly by zero on $A - T_{X_j}(1)$.

Writing $\tilde{\zeta}^r_j$ for such an extended vector field and $\tilde{\phi}^r_j : A \times \mathbb{R} \to A$ for its flow (which is obviously the identity off $T_{X_j}(1) \times \mathbb{R}$), we obtain then for every j the stratified homeomorphism of A

$$\tilde{f}_j = [\tilde{\phi}^1_j]_1 \circ \cdots \circ [\tilde{\phi}^{r_j}_j]_1 \;:\; A \longrightarrow A$$

extending the diffeomorphism f_{X_j} and which is the identity on $A - T_{X_j}(1)$.

Finally by recalling that the tubular neighbourhood $\{T_{X_j}\}_j$ may be chosen pairwise disjoint we find the claimed extension $\tilde{f} : A \to A$, of $f : \sqcup_j X_j \to \sqcup_j X_j$, on the whole of A, by gluing toghether all lifting $\tilde{f}_j$ i.e., by setting :

$$\tilde{f}(x) = \begin{cases} \tilde{f}_j(x) & \text{if } x \in \sqcup_j T_{X_j}(1); \\ \\ x & \text{if } x \in A - \sqcup_j T_{X_j}(1). \end{cases} \qquad \square$$

REMARK 2. The hypotheses of theorem 3 are verified for example when, as in theorem 2, f is given on a k-skeleton A_k of $\mathcal{X}$ and $A_{k-1} = \emptyset$ or again when all strata X_j on which f is defined are of maximal depth in $\mathcal{X}$ (see *i)* in theorem 2).

REMARK 3. Whitney stratifications and Bekka (c)-regular stratifications always admit a system of control data **[8]** and **[2]**, so these latter may be structured as abstract stratified sets. Hence the extension theorems 2 and 3 and Remark 2 hold for (b)- and (c)-regular stratifications.

PROPOSITION. *Theorems 2 and 3 hold again by considering for $\mathcal{X} = (A, \Sigma)$ a (w)-regular stratification (instead of an abstract stratified set).*

Proof. Notice that for $\mathcal{X}$ a (w)-regular stratification, the lifting of vector fields giving stratified continuous flows may be again obtained **[22]** (without using systems of control data). The proofs may then be obtained by slight modification of the proofs of theorems 2 and 3 in which we consider some arbitrary families of stratified tubular neighbourhoods $\{T_X\}_X$, of smooth projections $\{\pi_X : T_X \to X\}_X$, and distance functions $\{\rho_X : T_X \to [0, \infty[\}_X$ not necessarily caming from a system of control data. $\square$

Remark 4. Because Lipschitz regularity of a stratification implies (w)-regularity **[18]** then the extension theorems 2 and 3 hold again for Lipschitz stratifications.

Remark 5. With the same notations as in the proof of the weak extension theorem, suppose that $f_{|A_{k-1}} = 1_{A_{k-1}}$ is the identity map. As one can see through easy examples, *it is not true* in general that $f = \phi_1^1 \circ \cdots \circ \phi_1^s$ and $\lim_{x\to\partial S} f(x) = 1_{\partial S}(x)$ imply that for every $i = 1, \ldots, s$ the vector field ζ^i of ϕ_t^i tends to the zero vector field on $\partial S = \overline{S} - S \subseteq A_{k-1}$. However, it would seem reasonable to think that in the case where the diffeomorphism $f \in Diff_0(S, S)$ lies in a sufficiently small neighborhood of 1_S, one could improve the choice of all ϕ^i to have that their vector fields ζ^i verify the continuity $\lim_{x\to\partial S} \zeta^i(x) = 0$ $(\forall i = 1, \ldots, s)$ on ∂S.

In this case, an extension by 0 on the remaining part $A - T_S$ may be obtained for a vector field $\tilde{\zeta}^i$, which is a perturbation of ζ^i (slightly more complicated than in theorem 3), so that the *weak stratified extension theorem* would be improved giving a (global) *strong stratified extension theorem* of the diffeomorphism $f : S \to S$ on the whole stratified space A. On the other hand, we have to remark that the possibility of deducing that for all i, $\lim_{x\to\partial S} \zeta^i(x) = 0$, depends on an extension of McDuff's results (theorem 1 and corollary 1.3 **[14]**), which seems to us non trivial.

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