HORIZONTALLY-C^1 CONTROLLED STRATIFIED MAPS AND THOM'S FIRST ISOTOPY THEOREM

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Abstract. We state results concerning horizontally-C^1 regularity for a stratified map $f : \mathcal{X} \to \mathcal{X}'$. It requires the existence of local stratified (a)-regular horizontal foliations for $\mathcal{X}$ and $\mathcal{X}'$ (a smooth version of the conjectured complex analytic *"Whitney fibering"*) and implies a horizontally-C^1 version of Thom's first isotopy theorem. We also consider the more general notion of $\mathcal{F}$-semi-differentiability for f, $\mathcal{F}$ being a foliation, and state analogous theorems.

Applications stratifiées contrôlées horizontalement-C^1 et le premier théorème d'isotopie de Thom

Résumé. *On énonce des résultats sur la régularité horizontalement-C^1 pour un morphisme stratifié $f : \mathcal{X} \to \mathcal{X}'$. Elle dépend de l'existence de feuilletages horizontaux stratifiés (a)-réguliers de $\mathcal{X}$ et de $\mathcal{X}'$ (une version lisse de la "fibration de Whitney", conjecturée pour des ensembles analytiques complexes) et implique une version horizontalement-C^1 du premier théorème d'isotopie de Thom. On définit aussi la $\mathcal{F}$-semi-différentiabilité pour f, $\mathcal{F}$ étant un feuilletage, et énonce des théorèmes analogues.*

Version française abrégée. Etant donné des espaces stratifiés $\mathcal{X} = (A, \Sigma)$ et $\mathcal{X}' = (A', \Sigma')$ contenus respectivement dans des variétés lisses M et N, nous avons introduit dans **[7]** les notions de morphisme stratifié contrôlé $f : \mathcal{X} \to \mathcal{X}'$ *horizontalement-C^1*, et de *distribution canonique* $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ associée à chaque strate X de $\mathcal{X}$: un sous-fibré stratifié continu de TM dont les relèvements $\xi = \{\xi_Y\}_{Y \geq X}$ sur $\mathcal{D}_X$ des champs de vecteurs ξ_X définis sur X sont des extensions stratifiées continues canoniques de ξ_X.

On dit qu'un feuilletage $\mathcal{F} = \{ F_z \}_z$ d'un ouvert U de A, compatible avec les strates de $\mathcal{X}$, $C^{1,0}$ sur chaque strate, et de dimension $\leq \dim X$, est (a)*-regular sur une strate* $Y \geq X$ (resp. sur U) si et seulement si, pour toute suite $\{z_n\}_n \subseteq U$ telle que $\lim_n z_n = y \in Y \cap U$ (resp. $\forall Y \geq X$), $\lim_n T_{z_n} F_{z_n} \subseteq T_y Y$.

Pour toute strate X de $\mathcal{X}$ notons X' la strate de $\mathcal{X}'$ contenant $f(X)$. La régularité horizontalement-C^1 de f sur X signifie exactement que pour tout $Y > X$ (et donc $Y' \geq X'$), la restriction $f_{Y*|\mathcal{D}_{XY}}$ prolonge continûment la différentielle $f_{X*} : TX \to TX'$.

Les relèvements continus contrôlés de champs de vecteurs sont utiles pour étudier cette propriété : si un champ de vecteurs ξ_X est relevé de manière continu et contrôlé sur un voisinage T_X on a, sous l'hypothèse de l'existence d'un feuilletage horizontal local (a)-régulier, que le flot relevé est un prolongement horizontalement-C^1 du flot ϕ_X de ξ_X. Plus précisément:

THÉORÈME 1. *Soient $\mathcal{X}$ un espace stratifié (c)-régulier et X une strate de $\mathcal{X}$. S'il existe une distribution canonique $\mathcal{D}_X$ qui est involutive, alors pour tout champ de vecteurs C^1 ξ_X sur X, le flot $\phi = \{\phi_Y\}_{Y \geq X}$ (au temps t), du relèvement continu contrôlé $\{\xi_Y\}_{Y \geq X}$ de ξ_X sur $\mathcal{D}_X$, est horizontalement-C^1 sur X.*

On considére la trivialisation topologique locale d'une projection $\pi_X : T_X \to X$ (d'un système de données de contrôle pour $\mathcal{X}$ [**6**]), au-dessus d'un voisinage U_{x_0} de x_0 dans X:

$$H : \mathbb{R}^l \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0}) \quad , \quad H(t_1, \ldots, t_l, z_0) = \phi_l(t_l, \ldots, \phi_1(t_1, z_0)..)) \,.$$

Celle-ci s'obtient [**6**] par composition des flots ϕ_i des champs de vecteurs relevés contrôlés continus v_i dans le voisinage $\pi_X^{-1}(U_{x_0})$ de x_0 dans A, et définit un feuilletage horizontal local canonique $\mathcal{H}_{x_0} = \{M_{y_0} = H(\mathbb{R}^l \times y_0)\}_{y_0 \in \pi_X^{-1}(x_0)}$.

PROPOSITION. *Les conditions suivantes sont équivalentes:*

1) Pour chaque champ de vecteurs C^1 ξ_X sur U_{x_0}, le relèvement contrôlé $\eta = \{\eta_Y\}_{Y \geq X}$ tangent à $\mathcal{H}_{x_0}$ est continu sur U_{x_0}, et son flot $\psi = \cup_{Y \geq X} \psi_Y$ est horizontalement-C^1 sur U_{x_0}.

2) Pour chaque champ de vecteur coordonné E_i sur U_{x_0} le relèvement contrôlé w_i tangent à $\mathcal{H}_{x_0}$ est continu sur U_{x_0}, et son flot ψ_i est horizontalement-C^1 sur U_{x_0}.

3) Le feuilletage $\mathcal{H}_{x_0}$ est (a)-régulier sur U_{x_0} (c'est à dire qu'il vérifie une version lisse de la conjecture de fibration de Whitney [**12**]*).*

La régularité horizontalement-C^1 des flots des champs relevés acquiert donc un intérêt principal et la continuité des champs relevés contrôlés devient nécessaire ainsi que la (c)-régulité de $\mathcal{X}$ au sens de K. Bekka, car la (c)-régularité caractérise l'existence de relèvements de champs de vecteurs qui sont à la fois continus et contrôlés [**1**], [**2**], [**9**] [**10**]. Rappelons que les stratifications (b)-régulières de Whitney sont (c)-régulières [**1**].

THÉORÈME 2. *Soit $f : \mathcal{X} \to \mathcal{X}'$ un morphisme contrôlé entre espaces stratifiés (c)-réguliers admettant autour des points $x_0 \in X$ et $x'_0 = f(x_0) \in X'$ des feuilletages (a)-réguliers $\mathcal{H}_{x_0}$ et $\mathcal{H}_{x'_0}$ respectivement de $\pi_X^{-1}(U_{x_0})$ et de $\pi_{X'}^{-1}(U'_{x'_0})$. Si l'image via f de chaque feuille de $\mathcal{H}_{x_0}$ est contenue dans un unique feuille de $\mathcal{H}_{x'_0}$ alors f est horizontalement-C^1 sur U_{x_0}.*

Nous avons aussi une version horizontalement-C^1 du premier théorème d'isotopie de Thom:

THÉORÈME 3. *Soit $\mathcal{X}$ un espace stratifié (c)-régulier admettant un feuilletage horizontal local $\mathcal{H}_{x_0}$ de $\pi_X^{-1}(U_{x_0})$, (a)-régulier sur un voisinage U_{x_0} dans X d'un point $x_0 \in X$.*

Soit $f : \mathcal{X} \to M$ une submersion contrôlée propre à valeurs dans une variété M. Alors, pour tout voisinage coordonné W_{m_0} d'un point m_0 dans M, et pour tout voisinage V de x_0 vérifiant $\overline{V} \subset U_{x_0}$, $f^{-1}(W_{m_0})$ admet un homéomorphisme de trivialisation

$$H_{m_0} \quad : \quad W_{m_0} \times f^{-1}(m_0) \longrightarrow f^{-1}(W_{m_0})$$

horizontalement-C^1 sur $W_{m_0} \times [f^{-1}(m_0) \cap V]$ avec $H_{m_0}^{-1}$ horizontalement-C^1 sur $f^{-1}(W_{m_0}) \cap V$.

L'hypothèse d'existence de tels "bons" feuilletages rappelle la *conjecture de fibration de Whitney* ([**12**], §9) pour des variétés analytiques, partiellement démontrée par R. Hardt et D.

Sullivan [**5**]. En 1993 le deuxième auteur a énoncé la conjecture que *toute* stratification de Whitney C^1 possède localement un tel feuilletage (a)-régulier.

Dans leur livre récent [**10**] (III, 9.3), dans le but de démontrer que *"la multitransversalité est une condition suffisante pour la C^0-stabilité forte"*, A. du Plessis et C. T. C. Wall introduisent les rétractions *extrèmement adaptées* $(E - tame)$ $r : M \to N$ entre variétés lisses, et montrent qu'elles se caractérisent par le fait que les feuilletages définis par leurs fibres sont de classe $C^{0,1}$. Dans un contexte stratifié ces feuilletages $C^{0,1}$ sont des feuilletages (a)-réguliers et certains de nos résultats s'appliquent à la théorie de du Plessis et Wall [**8**].

Au §3 nous généralisons la régularité horizontalement-C^1 par la notion de $\mathcal{F}$-semi-différentiabilité de f, $\mathcal{F}$ étant un feuilletage local $C^{0,1}$, et nous donnons des analogues des théorèmes 1, 2 et 3 (théorèmes 4, 5 et 6).

1. Introduction. Given stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (A', \Sigma')$ in smooth manifolds M, resp. N, we introduced in [**7**] the notions of *horizontally-C^1* stratified controlled maps $f : \mathcal{X} \to \mathcal{X}'$, and *canonical distributions* $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ associated to each stratum X of $\mathcal{X}$. $\mathcal{D}_X$ is a continuous subbundle of TM, such that there are canonical stratified continuous extensions to $\mathcal{D}_X$ of vector fields defined on X.

For each stratum X of $\mathcal{X}$ let X' be the stratum of $\mathcal{X}'$ containing $f(X)$.

Horizontally-C^1 regularity of f on X means that $\forall\, Y > X$ (and $Y' \geq X'$), the restriction $f_{Y*|\mathcal{D}_{XY}} : \mathcal{D}_{XY} \to TY'$ extends continuously the differential $f_{X*} : TX \to TX'$.

Continuous controlled lifting of vector fields is particularly useful in studying such a property. In fact, if a vector field ξ_X is lifted to a stratified continuous (π, ρ)-controlled vector field $\xi = \{\xi_Y\}_{Y \geq X}$ on a neighborhood T_X of X in A, then assuming the existence of an involutive canonical distribution $\mathcal{D}_X$ or of an (a)-regular foliation, the lifted flow $\phi = \cup_{Y \geq X} \phi_Y$ on T_X is a horizontally-C^1 extension of ϕ_X (theorem 1).

To find "good" foliations we consider a topological trivialisation in a neighbourhood U_{x_0} of a point x_0 in X of a projection $\pi_X : T_X \to X$ associated to a system of control data [**6**]:

$$H : \mathbb{R}^l \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0}) \quad , \quad H(t_1, \ldots, t_l, z_0) = \phi_l(t_l, \ldots, \phi_1(t_1, z_0)..)) \, .$$

This is obtained [**6**] by composition of the flows ϕ_i of *continuous* lifted controlled vector fields v_i in $\pi_X^{-1}(U_{x_0})$, and defines a stratified horizontal foliation $\mathcal{H}_{x_0} = \{H(\mathbb{R}^l \times y_0)\}_{y_0 \in \pi_X^{-1}(x_0)}$ near x_0 which is (a)-regular (see definition 1) iff the ϕ_i are horizontally-C^1 (proposition 2, §2). Thus horizontally-C^1 regularity of flows of continuous liftings of vector fields becomes of principal interest. Continuity of the controlled lifting of vector fields is necessary, as is (c)-regularity of $\mathcal{X}$ in the sense of K. Bekka, because (c)-regularity characterizes the existence of lifting of vector fields which are simultaneously continuous and controlled [**1**], [**2**], [**9**], [**10**]. Recall that Whitney (b)-regular stratifications are (c)-regular [**1**].

For more general morphisms $f : \mathcal{X} \to \mathcal{X}'$, theorem 2 says that horizontally-C^1 regularity of f depends on the existence of a local (a)-regular horizontal foliation, and implies a horizontally-C^1 Thom isotopy theorem (theorem 3).

The existence of a local (a)-regular horizontal foliation recalls the *Whitney fibering conjecture* [**12**] for analytic varieties, which was partially proved by Hardt and Sullivan [**5**], and is strongly related to the theory of E-tame retractions of A. du Plessis and C. T. C. Wall [**10**].

We generalise horizontally-C^1 regularity by introducing $\mathcal{F}$-semi-differentiability of f, for $\mathcal{F}$ a stratified $C^{0,1}$ foliation, and give analogues of theorems 1, 2, and 3 (theorems 4, 5 and 6).

2. Horizontally-C^1 controlled flows of continuous liftings of vector fields. Fix a (c)-regular stratified space $\mathcal{X}$, a stratum X of $\mathcal{X}$ and a vector field ξ_X on X having a global flow $\phi_X : X \times \mathbb{R} \to X$. Denote by $\xi = \{\xi_Y\}_{Y \geq X}$ the continuous controlled lifting of ξ_X tangent to a canonical distribution $\mathcal{D}_X$ ([**7**], theorem 2) and denote by $\Phi : T^{\epsilon}_X \times \mathbb{R} \to T^{\epsilon}_X$ its flow (with the usual notations), which exists by the control conditions, and is globally defined on a stratified neighbourhood $T^{\epsilon}_X = \cup_{Y \geq X} T^{\epsilon}_{XY}$, $(T^{\epsilon}_{XY} = T^{\epsilon}_X \cap Y)$ of X. Then it is well-known that:

i) Φ *is continuous* (by the control conditions);

ii) Φ *is* C^1 *with respect to* t (by the continuity of Φ and ξ and because $\frac{\partial \Phi}{\partial t} = \xi \circ \Phi$);

iii) Φ *is not in general* C^1 (counterexample of the *"four lines family"* [**12**], [**4**]).

In what follows, we replace $\mathcal{X}$ by the stratified neighbourhood $T^{\epsilon}_X = \cup_{Y \geq X} T^{\epsilon}_{XY}$; then each T^{ϵ}_{XY} is identified with Y. We write $\phi = \Phi_t$ for the flow at a time t and $\phi_Y = \Phi_{Yt}$. Then $\phi = \cup_{Y \geq X} \phi_Y$ is a stratified homeomorphism ([**6**], [**1**]).

<u>The case of involutive $\mathcal{D}_X$.</u>

Theorem 1. *If* $\mathcal{D}_X$ *is involutive, the flow* ϕ *is horizontally-*C^1 *on* X.

Remark 1. If $\dim X = 1$, or if $\dim X = \dim \mathcal{X} - 1$, then every $\mathcal{D}_X$ is involutive.

Remark 2. In 1994 A. du Plessis and the second author found that even for a Whitney (b)-regular stratification not every canonical distribution $\mathcal{D}_X$ is involutive. The second author conjectured in 1993 that every (b)-regular stratification admits locally *some* involutive canonical distribution $\mathcal{D}_X$; this remains unsolved, and could even be true for (c)-regular stratifications.

<u>The general case.</u> Let $x_0 \in X$, U a neighbourhood of x_0 in A and $\mathcal{F} = \{ F_z \}_z$ a foliation of dimension $h \leq \dim X$ of U compatible with the strata of $\mathcal{X}$ and which is $C^{1,0}$ on each stratum.

Definition 1. We will say that $\mathcal{F}$ is (a)*-regular* on a stratum $Y \geq X$ (resp. on U) iff for every sequence $\{z_n\}_n \subseteq U$ such that $\lim_n z_n = y \in Y \cap U$ (resp. $\forall Y \geq X$) we also have $\lim_n T_{z_n} F_{z_n} \subseteq T_y Y$. When $\dim \mathcal{F} = \dim X$, the existence of such a regular foliation recalls a conjecture of Whitney for (b)-regular stratifications of analytic varieties and can be interpreted as a smooth version of *Whitney fibering* ([**12**], §9). By Theorem 1 in [**7**] we have :

Proposition 1. *If there is a neighbourhood* U_{x_0} *in* X *of a point* $x_0 \in X$ *such that a canonical distribution* $\mathcal{D}_X$ *is involutive on* $\pi_X^{-1}(U_{x_0})$*, then the horizontal foliation* $\mathcal{H}_{x_0}$ *is* (a)*-regular on* $\pi_X^{-1}(U_{x_0})$*. Conversely if the foliation* $\mathcal{H}_{x_0}$ *is* (a)*-regular on* $\pi_X^{-1}(U_{x_0})$*, then* $\mathcal{D}' = T\mathcal{H}_{x_0}$ *defines an involutive local canonical distribution on* $\pi_X^{-1}(U_{x_0})$*.*

When $\mathcal{D}_X$ is possibly not involutive, we can again work on a trivialised stratified space $\pi_X^{-1}(U_{x_0})$, and replace $\mathcal{D}_X$ by the distribution $T\mathcal{H}_{x_0}$ tangent to the local foliation $\mathcal{H}_{x_0}$. Instead of the lift ξ of ξ_X to $\mathcal{D}_X$, we can consider the unique lifting $\eta = \{\eta_Y\}_{Y \geq X}$ tangent to the foliation $\mathcal{H}_{x_0}$, defined on $\pi_X^{-1}(U_{x_0})$. If $\mathcal{D}_X$ is not involutive η and its flow Ψ *do not coincide* with ξ and Φ. In such a situation, for t small enough so that $\Psi_{X,t}(x_0) \in U_{x_0}$, we have :

Proposition 2. *The following conditions are equivalent:*

1) For each C^1 *vector field* ξ_X *on* U_{x_0} *the controlled lifting* $\eta = \{\eta_Y\}_{Y \geq X}$ *tangent to* $\mathcal{H}_{x_0}$ *is continuous, and has a horizontally-*C^1 *flow* $\psi = \cup_{Y \geq X} \psi_Y$*, on* U_{x_0}*.*

2) For each coordinate vector field E_i of U_{x_0} the controlled lifting w_i tangent to $\mathcal{H}_{x_0}$ is continuous, and has a horizontally-C^1 flow ψ_i, on U_{x_0}.

3) $\mathcal{H}_{x_0}$ is (a)-regular on U_{x_0} (so verifies a smooth version of Whitney's fibering conjecture).

For more general stratified maps (not lifted flows) we prove the following theorem :

Theorem 2. *Let $f : \mathcal{X} \to \mathcal{X}'$ be a controlled map between (c)-regular stratified spaces having near $x_0 \in X$ and $x'_0 = f(x_0) \in X'$ (a)-regular foliations $\mathcal{H}_{x_0}$ of $\pi_X^{-1}(U_{x_0})$ and $\mathcal{H}_{x'_0}$ of $\pi_{X'}^{-1}(U'_{x'_0})$. If the image of each leaf of $\mathcal{H}_{x_0}$ lies in a leaf of $\mathcal{H}_{x'_0}$, f is horizontally-C^1 on U_{x_0}.*

Using theorem 2 we obtain a horizontally-C^1 version of the Thom's first isotopy theorem :

Theorem 3. *Let $\mathcal{X}$ be a (c)-regular stratification admitting a local horizontal foliation $\mathcal{H}_{x_0}$ of $\pi_X^{-1}(U_{x_0})$, (a)-regular on $U_{x_0} \subseteq X$, and let $f : \mathcal{X} \to M$ be a proper submersion into a manifold M. For every coordinate neighbourhood W_{m_0} of $m_0 \in M$, and neighbourhood V of x_0 satisfying $\overline{V} \subset U_{x_0}$, the fiber $f^{-1}(W_{m_0})$ admits a trivialising homeomorphism*

$$H_{m_0} \quad : \quad W_{m_0} \times f^{-1}(m_0) \longrightarrow f^{-1}(W_{m_0})$$

horizontally-C^1 on $W_{m_0} \times [f^{-1}(m_0) \cap V]$ with $H_{m_0}^{-1}$ horizontally-C^1 on $f^{-1}(W_{m_0}) \cap V$.

Remark 3. The trivialisation H_{m_0} may have non-bounded differentials along the fibers of f. So, by Theorem 3 of **[7]**, the horizontally-C^1 regularity obtained for H_{m_0} in the previous theorem is best possible. It remains an open problem as to whether all (c)-regular stratifications admit local (a)-regular horizontal foliations. It is also unknown for (b)-regular stratifications, even in the semialgebraic or subanalytic cases.

Remark 4. In their recent book **[10]** (III, 9.3), with the aim of showing that *"multi-transversality implies C^0 strong stability"*, A. du Plessis and C. T. C. Wall introduce *$E-$tame retractions* $r : M \to N$ between smooth manifolds, and prove that they are characterised by the property that their fibres define a foliation of class $C^{0,1}$. In a stratified context such $C^{0,1}$ foliations are equivalent to (a)-regular foliations and some of our results apply to the du Plessis-Wall theory **[8]**. Future work of the authors will study further consequences of having locally such regular foliated structure.

3. $\mathcal{F}$-semidifferentiability. We introduce here $\mathcal{F}$-semidifférentiability, a regularity condition which generalizes semidifférentiability **[8]** and refines horizontally-C^1 regularity.

Definition 2. Let $\mathcal{F} = \{F_z\}_z$ be an (a)-regular stratified $C^{1,0}$ foliation of an open set U of A. We say a morphism $f : \mathcal{X} \to \mathcal{X}'$ is *$\mathcal{F}$-semidifférentiable at $y \in U$* iff for every $v \in T_yY$ and sequence $\{(z_n, v_n)\}$ tangent to $\mathcal{F}$ and converging to (y, v), we have $\lim_n f_{Z_n * z_n}(v_n) = f_{Y*y}(v)$, where Z_n denotes the stratum containing z_n : the differentials of $f_{|F_{z_n}}$ must converge to the differential of $f_{|F_y}$.

The results of section 2 hold again for $\mathcal{F}$-semidifférentiability :

Theorem 4. *Consider a (c)-regular stratified space $\mathcal{X}$, a stratum X of $\mathcal{X}$ and a C^1 vector field ξ_X on X. If $\mathcal{H}_{x_0} = \{H(y_0 \times \mathbb{R}^l)\}_{y_0 \in \pi_X^{-1}(x_0)}$ is (a)-regular then the controlled lifted flow $\phi = \cup_{Y \geq X} \phi_Y$ is $\mathcal{H}_{x_0}$-semidifferentiable on $\pi_X^{-1}(U_{x_0})$.*

Remark 5. If $\dim X \in \{1, \dim \mathcal{X} - 1\}$, the foliation $\mathcal{H}_{x_0}$ induced by every canonical distribution $\mathcal{D}_X$ is (a)-regular.

By considering more general stratified maps f we find analogues of theorems 2 and 3:

THEOREM 5. *Let $f : \mathcal{X} \to \mathcal{X}'$ be a controlled map between (c)-regular stratified spaces and $\mathcal{H}$ and $\mathcal{H}'$ be two (a)-regular stratified foliations of $\pi_X^{-1}(U_{x_0})$ and $\pi_{X'}^{-1}(U'_{x'_0})$. If f sends each leaf of $\mathcal{H}$ into a unique leaf of $\mathcal{H}'$, f is $\mathcal{H}$-semidifferentiable.*

THEOREM 6. *Let $\mathcal{X}$ be a (c)-regular stratified space, X a stratum, $x_0 \in X$, and $\mathcal{H}$ an (a)-regular foliation of $\pi_X^{-1}(U_{x_0})$. Let $f : \mathcal{X} \to M$ be a proper stratified submersion into a manifold M. For each coordinate neighbourhood W_{m_0} of m_0 in M and each neighbourhood V of x_0 with $\overline{V} \subseteq U_{x_0}$, there is a $W_{m_0} \times \mathcal{H}\big|_{f^{-1}(m_0)\cap\pi_X^{-1}(V)}$-semidifferentiable stratified homeomorphism $H : W_{m_0} \times f^{-1}(m_0) \to f^{-1}(W_{m_0})$ whose inverse H^{-1} is $\mathcal{H}\big|_{f^{-1}(W_{m_0})\cap\pi_X^{-1}(V)}$-semidifferentiable.*

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