SEMI-DIFFERENTIABLE STRATIFIED MORPHISMS

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Abstract. Near each stratum of a stratified space we use canonical distributions to improve known theorems on continuous controlled lifting of vector fields. We define semi-differentiable and horizontally-C^1 stratified controlled morphisms $f : \mathcal{X} \to \mathcal{X}'$ between stratified spaces. When $\mathcal{X}'$ is a smooth manifold, f is semi-differentiable. Otherwise semi-differentiability is equivalent to having bounded derivatives and being horizontally-C^1. A semi-differentiable stratified homeomorphism preserves regularity of stratified subspaces.

Morphismes stratifiés semi-différentiables

Résumé. *Autour de chaque strate d'un espace stratifié, on définit des distributions canoniques afin d'améliorer les théorèmes connus sur les relèvements continus controlés de champs de vecteurs. Nous introduisons les morphismes stratifiés contrôlés $f : \mathcal{X} \to \mathcal{X}'$ horizontalement-C^1, et semi-différentiable entre deux espaces stratifiés. Si $\mathcal{X}'$ est une variété lisse, f est semi-différentiable. En général, la semi-différentiabilité équivaut à ce que f ait toutes ses dérivées bornées et soit horizontalement-C^1. Un homéomorphisme stratifié semi-différentiable préserve la régularité de sous-espaces stratifiés.*

Version française abrégée. Appelons $\mathcal{X} = (A, \Sigma)$ un *espace stratifié* dans une variété C^1 M, quand A est un fermé de M muni d'une *stratification* Σ : une partition localement finie en sous-variétés C^1 connexes (les *strates*) vérifiant la condition de la frontière **[3], [7], [8], [19]**.

Nous améliorons les théorèmes de K. Bekka **[1], [2]** et A. du Plessis **[14]** sur les prolongements continus de champs de vecteurs stratifiés par la notion de *distribution canonique* $\mathcal{D}_X$: un sous-fibré continu de TM associé à chaque strate X tel que les relevés des champs de vecteurs sur $\mathcal{D}_X$ soient des extensions stratifiées canoniques continues de champs de vecteurs sur X.

THÉORÈME. *Soient $\mathcal{X}$ un espace stratifié (c)-régulier dans une variété C^1 M et $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times [0, \infty[\,\}_{X \in \Sigma}$ un système de données de contrôle de $\mathcal{X}$.*

Pour toute strate X de $\mathcal{X}$, il existe une distribution stratifiée continue $\mathcal{D}_X : T_X^\epsilon \to \mathbb{G}_n^l$ où $l = \dim X$, $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ et $\forall Y \geq X$, $\mathcal{D}_{XY} = \mathcal{D}_{XY|T_{XY}}$ avec $T_{XY} = T_X \cap Y$, telle que :

i) $\forall Y \geq X$ la restriction $\mathcal{D}_{XY}$ est un sous-fibré de $\ker \rho_{XY*}$;

ii) $\mathcal{D}_{XX}(x) = T_xX \ , \ \forall x \in X;$
iii) $T_yY = \mathcal{D}_{XY}(y) \oplus \ker \pi_{XY*y}$ *est une somme directe* $\forall\, y \in T^\epsilon_{XY}$;
iv) la restriction $\pi_{XY*y|} : \mathcal{D}_{XY}(y) \to T_xX$ *où* $x = \pi_{XY}(y)$ *est un isomorphisme;*
v) pour tout champ de vecteurs C^1 ξ_X *sur* X*, la formule :*

$$\xi_Y(y) = \mathcal{D}_{XY}(y) \cap \pi^{-1}_{XY*y}(\xi_X(x)) \quad , \quad x = \pi_X(y)$$

définit un relèvement continu contrôlé $\xi = \{\xi_Y\}_{Y \geq X}$ *de* ξ_X *sur* $T^\epsilon_X = \cup_{Y\geq X} T^\epsilon_{XY}$.

Ensuite nous introduisons la *semi-différentiabilité* pour des morphismes stratifiés. Soit $f : \mathcal{X} \to \mathcal{X}'$ un morphisme stratifié continu, et pour toute strate X de $\mathcal{X}$, notons X' l'unique strate de $\mathcal{X}'$ contenant $f(X)$. La semi-différentiabilité de $f : \mathcal{X} \to \mathcal{X}'$ signifie que $\forall\, Y > X$ (et donc $Y' \geq X'$), la différentielle $f_{Y*} : TY \to TY'$ de la restriction $f_Y : Y \to Y'$ prolonge continûment la différentielle $f_{X*} : TX \to TX'$ de $f_X : X \to X'$.

Nous considérons des morphismes stratifiés $f : \mathcal{X} \to \mathcal{X}'$ contrôlés **[3]**, **[7]**, **[8]** par rapport à deux systèmes de données de contrôle $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times \mathbb{R}\}_{X\in\Sigma}$ de $\mathcal{X}$ et $\mathcal{F}' = \{(\pi_{X'}, \rho_{X'}) : T_{X'} \to X'\}_{X'\in\Sigma'}$ de $\mathcal{X}'$ où T_X est un voisinage tubulaire de la strate X, π_X est projection sur X et ρ_X est une fonction telle que $X = \rho_X^{-1}(0)$ et $(\pi_X, \rho_X)|Y : T_X \cap Y \to X \times \mathbb{R}$ est une submersion $\forall\, Y > X$.

Un morphisme contrôlé $f : \mathcal{X} \to M$ à valeurs dans une variété C^1 est semi-différentiable. En général, la semi-différentiabilité nécessite que les différentielles $g_{XY}(y) = ||f_{Y*y|\ker \pi_{XY*y}}||$ de f le long des fibres des projections $\pi_{XY} : T_{XY} = T_X \cap Y \to X$ soient bornées près de X. Nous introduisons la régularité *horizontalement-*C^1 en séparant la convergence *horizontale* de la convergence *verticale* dans la limite $\lim\limits_{(y,v)\to(x,u)} f_{Y*y}(v) = f_{X*x}(u)$. On obtient :

Théorème. *Un morphisme stratifié contrôlé* f *est semi-différentiable en* $x \in X$ *si et seulement si* f *est horizontalement-*C^1 *en* x *et s'il existe un voisinage* U_x *de* x *dans* A *tel que pour toute strate* $Y > X$*,* $g_{XY}(y) = ||f_{Y*y|\ker \pi_{XY*y}}||$ *soit bornée sur* $U_x \cap Y$*.*

La semi-différentiabilité apporte une réponse au problème suivant : *"donner des conditions suffisantes pour qu'un homéomorphisme stratifié f (difféomorphisme sur chaque strate) préserve la régularité des sous-espaces stratifiés"*. Les auteurs de cet article, avec A. du Plessis, prouvent dans **[11]** (voir aussi **[13]**) le théorème de transversalité suivant qui améliore ceux de C. McCrory **[9]**, **[10]** et M. Goresky **[4]**.

Théorème. *Soient* $\mathcal{X} = (A, \Sigma)$ *un espace stratifié* (c)*-régulier dans une variété* C^∞ M *et* $g : \mathcal{Y} \to \mathcal{X}$ *un morphisme stratifié défini sur un espace stratifié arbitraire* $\mathcal{Y}$*.*

Pour tout objet sous-stratifié $\mathcal{W}$ *et pour chaque voisinage ouvert* U *de* W *dans* A *il existe une isotopie stratifiée* $\Phi_t : \mathcal{X} \to \mathcal{X}$ *telle que la déformation de* $\mathcal{W}$ *au temps* $t = 1$*,* $\mathcal{W}' = \Phi_1(\mathcal{W})$ *est transverse à* g *et* $W' \subseteq U$*. De plus, si* Φ_1 *est semi-différentiable on a :*

i) $\mathcal{W}$ (c)*-régulier* $\Rightarrow$ $\mathcal{W}'$ *est* (c)*-régulier;*
ii) $\mathcal{W}$ (a)*-régulier* $\Rightarrow$ $\mathcal{W}'$ *est* (a)*-régulier.*

Ce théorème joue un rôle essentiel dans les théories $BH_*(\mathcal{X})$ et $BH^*(\mathcal{X})$ de représentation d'homologie et cohomologie d'un espace stratifié (c)-régulier $\mathcal{X}$ par des cycles (c)-réguliers **[13]** analogues aux theories $WH_*(X)$ et $WH^*(X)$ de Goresky **[4]** pour une stratification de Whitney.

Dans **[12]** nous donnons des conditions suffisantes pour que soit horizontalement-C^1 et montrons que ceci dépend d'une régularité feuilletée de $\mathcal{X}$ et $\mathcal{X}'$, interprétée comme une version lisse de la *"fibration de Whitney"* (**[20]** §9); une conjecture de Whitney est qu'une telle fibration existe pour une variété analytique complexe - R. Hardt et D. Sullivan ont donné une réponse partielle [**6**]. Le second auteur conjecture que toute stratification de Whitney possède la régularité feuilletée. Nous donnons aussi une version horizontalement-C^1 du premier théorème d'isotopie de Thom [**12**].

1. Introduction. We call $\mathcal{X} = (A, \Sigma)$ a *stratified space* in a manifold M, when A is a closed subset of M and Σ is a *stratification* of A : a locally finite partition of A into connected C^1 submanifolds (*strata*) satisfying the frontier condition [**3**], [**7**], [**8**], [**19**].

In this paper, we improve results of K. Bekka [**1**], [**2**] and A. du Plessis [**14**], introducing the notion of *canonical distribution* $\mathcal{D}_X$: a continuous subbundle of TM, associated to each stratum X of $\mathcal{X}$, such that lifting to $\mathcal{D}_X$ one finds canonical stratified continuous extensions of vector fields defined on X. Then we introduce *semi-differentiability* for a stratified map $f : \mathcal{X} \to \mathcal{X}'$.

We call *stratified morphism* a continuous map $f : \mathcal{X} \to \mathcal{X}'$ between stratified spaces which sends each stratum X into a unique stratum X' such that each restriction $f_X : X \to X'$ is C^1.

Semi-differentiability is based on the idea that $f : \mathcal{X} \to \mathcal{X}'$ is sufficiently regular if, $\forall Y > X$ (and so $Y' \geq X'$), the differential $f_{Y*} : TY \to TY'$ of the restriction $f_Y : Y \to Y'$ extends continuously the differential $f_{X*} : TX \to TX'$ of the restriction f_X.

We consider stratified morphisms $f : \mathcal{X} \to \mathcal{X}'$ which are controlled [**3**], [**7**], [**8**] with respect to control data $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times \mathbb{R}\}_{X \in \Sigma}$ for $\mathcal{X}$ and $\mathcal{F}' = \{(\pi_{X'}, \rho_{X'}) : T_{X'} \to X'\}_{X' \in \Sigma'}$ for $\mathcal{X}'$ where T_X is a tubular neighbourhood of the stratum X, π_X is projection on X, and ρ_X is a function such that $X = \rho_X^{-1}(0)$ and $(\pi_X, \rho_X)|Y : T_X \cap Y \to X \times \mathbb{R}$ is a submersion $\forall Y > X$.

A controlled morphism $f : \mathcal{X} \to M$ into a C^1 manifold M is semi-differentiable. Otherwise, semi-differentiability requires the boundedness of differentials $g_{XY}(y) = ||f_{Y*y|\ker \pi_{XY*y}}||$ of f along the fibres of the restricted projections $\pi_{XY} : T_{XY} = T_X \cap Y \to X$ near X. We introduce the weaker notion of *horizontally-C^1 stratified morphism*, obtained from semi-differentiability by separating *horizontal* and *vertical convergence* in the limit $\lim_{(y,v)\to(x,u)} f_{Y*y}(v) = f_{X*x}(u)$. Semi-differentiability of f is equivalent to f being horizontally-C^1 and the g_{XY} being bounded.

Semi-differentiability provides an answer to the following problem : *"find sufficient conditions for a stratified homeomorphism (C^1 diffeomorphism on each stratum) to preserve regularity of stratified subspaces"*. With A. du Plessis we prove [**11**], [**13**] a theorem of stratified transversality (improving results of McCrory [**9**], [**10**] and Goresky [**4**]) in which semi-differentiability is used to preserve (a) and (c)-regularity of stratified subspaces (Theorem 4).

The interest of horizontally-C^1 regularity will be clarified in [**12**] where we give sufficient conditions for $f : \mathcal{X} \to \mathcal{X}'$ to be horizontally-C^1, showing that this depends on a certain foliated regularity of $\mathcal{X}$ and $\mathcal{X}'$ which may be interpreted as a smooth version of *"Whitney fibering"* [**20**], conjectured by Whitney to exist in the complex analytic case. R. Hardt et D. Sullivan have proved a weakened version of Whitney's conjecture [**6**]. The second author conjectures that every C^1 Whitney stratification possesses this foliated regularity. We also present in [**12**] a horizontally-C^1 version of Thom's first isotopy theorem.

2. Stratified continuous lifting of vector fields. Most proofs of Thom's first isotopy theorem use stratified vector fields. The next theorem ([**1**], [**2**], [**14**], and [**15**], p.42) says that K. Bekka's (c)-regularity characterizes continuous controlled lifting to T_X of each C^1 vector field on X. In 1984 M. Shiota [**16**] announced that continuous controlled liftings exist for (b)-regular stratifications with a proof of the 2-strata case; more details are in his book (Lemma I.1.5 of [**16**]). A. du Plessis has given a proof of the general case [**14**]. Bekka observed in 1988 ([**1**], [**2**]) that (c)-regularity, implied by (b)-regularity, is enough. S. Simon used this lifting property to prove a Poincaré-Hopf theorem for totally radial stratified vector fields [**18**], while H. Hamm [**5**] has recently used it to give a simplified proof of the Fundamental Theorem of Goresky-MacPherson's stratified Morse theory.

THEOREM 1. (Bekka-du Plessis). *A stratification Σ of $A \subseteq \mathbb{R}^n$ is (c)-regular if and only if every C^1 vector field ξ_X on a stratum X may be extended to a stratified continuous controlled vector field ξ in a neighbourhood T_X of X, of class C^1 on each stratum.*

By similar methods to those of Shiota, Bekka and du Plessis we improve the previous theorem and obtain a continuous lifting of the Gauss map $\mathcal{D}_{XX} : X \to \mathbb{G}_n^l$, $\mathcal{D}_{XX}(x) = T_xX$, giving the following natural formulation of continuous controlled lifting [**13**].

THEOREM 2. *Let $\mathcal{X}$ be a (c)-regular stratified set in a C^1 manifold M and fix a system of control data $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times [0, \infty[\,\}_{X \in \Sigma}$ of $\mathcal{X}$.*

For every stratum X of $\mathcal{X}$, there exists a stratified continuous distribution $\mathcal{D}_X : T_X^\epsilon \to \mathbb{G}_n^l$ where $l = \dim X$, $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ and $\mathcal{D}_{XY} = \mathcal{D}_{X|T_{XY}}$, $\forall Y \geq X$ such that :

i) $\forall Y \geq X$, $\mathcal{D}_{XY}$ defines a subbundle of $\ker \rho_{XY*}$;
ii) $\mathcal{D}_{XX}(x) = T_xX$, $\forall x \in X$;
iii) $T_yY = \mathcal{D}_{XY}(y) \oplus \ker \pi_{XY*y}$, $\forall\, y \in T_{XY}^\epsilon$;
iv) the restriction $\pi_{XY*y|} : \mathcal{D}_{XY}(y) \to T_xX$, where $x = \pi_{XY}(y)$, is an isomorphism;
v) for every C^1 vector field ξ_X on X, the formula

$$\xi_Y(y) = \mathcal{D}_{XY}(y) \cap \pi_{XY*y}^{-1}(\xi_X(x)) \quad , \quad x = \pi_{XY}(y)$$

defines a continuous controlled lifting $\xi = \{\xi_Y\}_{Y \geq X}$ of ξ_X on $T_X^\epsilon = \cup_{Y \geq X} T_{XY}^\epsilon$.

DEFINITION 4. The distribution $\mathcal{D}_X$ depends on a partition of unity used to glue distributions not globally defined. However we refer to $\mathcal{D}_X : T_X \to \mathbb{G}_n^l$ as a *(global) canonical distribution of X on A*. For every neighbourhood U_{x_0} of a point x_0 on A, we call *local canonical distribution* every global canonical distribution of $U_{x_0} \cap X$ on $U_{x_0} \cap A$.

3. Semi-differentiability and horizontally-C^1 stratified morphisms. In this section we consider stratified morphisms f between stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (A', \Sigma')$ whose supports A and A' are contained in C^1 manifolds M and N, and whose stratifications Σ and Σ' satisfy at least Whitney (a)-regularity [**20**], and the frontier condition.

Suppose that the stratifications Σ and Σ' are equipped with systems of control data $\mathcal{F}$ and $\mathcal{F}'$ [**3**], [**7**], [**8**], then f is *controlled* (with respect to $\mathcal{F}$ and $\mathcal{F}'$) iff the following *control conditions* hold : $\pi_{X'} f_Y = f_X \pi_X$ and $\rho_{X'} f_Y = \rho_X$, $\forall\, X < Y \in \Sigma$.

DEFINITION 1. We say that a stratified morphism $f : \mathcal{X} \to \mathcal{X}'$ is *semi-differentiable at x* of $X \in \Sigma$ if for each stratum $Y > X$ (i.e. $\overline{Y} \supseteq X$) and for each sequence $\{(y_n, v_n)\}_n$ in the tangent space $TY = \cup_{y \in Y} \{y\} \times T_yY$,

$$\text{“} \lim_{n\to\infty} (y_n, v_n) = (x, v) \in TX \quad \Rightarrow \quad \lim_{n\to\infty} f_{Y*y_n}(v_n) = f_{X*x}(v)\text{''} .$$

This makes sense because by the frontier condition, $X \subseteq \overline{Y} \subseteq M$, and (a)-regularity implies that $TX \subseteq \overline{TY}$ and $TX' \subseteq \overline{TY'}$ in TM and TN.

We say f is *semi-differentiable on a stratum* $X \in \Sigma$ iff it is semi-differentiable at every $x \in X$ and that f is *semi-differentiable* iff it is semi-differentiable on every stratum $X \in \Sigma$.

Remark 1. Semi-differentiability at x is weaker than C^1-differentiability of f at x.

Proposition 1. *Let $f : \mathcal{X} \to M$ be a stratified morphism into a C^1 manifold M. If there is a system of control data $\mathcal{F}$ on $\mathcal{X}$ whose projections $\{\pi_X : T_X \to X\}_{X\in\Sigma}$ are smooth and with respect to which f is controlled, then f is semi-differentiable.*

Remark 2. Every (b)- or (c)-regular stratification admits a system of control data whose projections and distance functions are C^1 maps (**[3], [7], [1]**). Clearly when one of $\mathcal{X}$ and $\mathcal{X}'$ is not embedded in a manifold (as may occur for abstract stratified sets **[7], [8]**), then differentiability of π_X and semi-differentiability of $f : \mathcal{X} \to M$ have no meaning.

Semi-differentiability at $x \in X$ requires boundedness of the differentials $f_{Y*y|\ker\pi_{XY*y}}$: $\ker\pi_{XY*y} \to \ker\pi_{X'Y'*y'}$ along fibers of the projections $\pi_{XY} : T_{XY} \to X$ near x :

Proposition 2. *If there exists a pair of strata $X < Y \in \Sigma$ such that the function $g_{XY}(y) = ||f_{Y*y|\ker\pi_{XY*y}}||$ is not bounded near x in $X \cup Y$, then f is not semi-differentiable.*

Proposition 2 suggests weakening the semi-differentiability condition by separating *horizontal* and *vertical convergence* of the limit $\lim_{(y,v)\to(x,u)} f_{Y*y}(v) = f_{X*x}(u)$.

Definition 2. A stratified morphism $f : \mathcal{X} \to \mathcal{X}'$ is *horizontally-C^1 at a point x* of a stratum $X \in \Sigma$ if there exists a local canonical distribution $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y\geq X}$ such that the limit condition in the definition of semi-differentiability holds for vectors $v_n \in T_{y_n}Y$ which are *horizontal*, meaning that $v_n \in \mathcal{D}_{XY}(y_n)$. Vectors $v_n \in \ker\pi_{XY*y_n}$ are called *vertical.* Thus f is horizontally-C^1 at $x \in X$ iff $\forall\, Y > X$ the restriction of the differential to a canonical distribution $f_{X*} \cup f_{Y*|\mathcal{D}_{XY}} : TX \cup \mathcal{D}_{XY} \to TX' \cup TY'$ is continuous at x. We say that f is *horizontally-C^1 on a stratum* X (resp. *on the whole of* $\mathcal{X}$) when it is horizontally-C^1 at each point $x \in X$ (resp. on each stratum X).

Theorem 3. *A controlled stratified morphism f is semi-differentiable at $x \in X$ if and only if it is horizontally-C^1 at x and there exists a neighbourhood U_x of x in A such that for every stratum $Y > X$ the function $g_{XY}(y) = ||f_{Y*y|\ker\pi_{XY*y}}||$ is bounded in $U_x \cap Y$.*

The original motivation of semi-differentiability was to find sufficient conditions for a stratified transversalising isotopy to preserve regularity of stratified subspaces. We show in **[11]** the following theorem, using a delicate extension theorem for stratified homeomorphisms:

Theorem 4. *Let $\mathcal{X} = (A, \Sigma)$ be a (c)-regular stratified space, with A a closed subset of a C^∞ manifold M and let $g : \mathcal{Y} \to \mathcal{X}$ be a stratified morphism defined on a stratified space $\mathcal{Y}$.*

For each substratified object $\mathcal{W}$ of $\mathcal{X}$ and each open neighbourhood U of W in A there exists a stratified isotopy $\Phi_t : \mathcal{X} \to \mathcal{X}$ such that the deformation of $\mathcal{W}$ at time $t = 1$, $\mathcal{W}' = \Phi_1(\mathcal{W})$ is transverse to g, and $W' \subseteq U$. If moreover Φ_1 is semi-differentiable, then :

i) $\mathcal{W}$ (c)-regular $\Rightarrow$ $\mathcal{W}'$ is (c)-regular;

ii) $\mathcal{W}$ (a)-regular $\Rightarrow$ $\mathcal{W}'$ is (a)-regular.

This theorem plays an essential role in the theory of geometric representation of homology and cohomology ($BH_*(\mathcal{X})$ et $BH^*(\mathcal{X})$) of a (c)-regular stratified space $\mathcal{X}$ by (c)-regular cycles **[13]**, analogous to the theory of Goresky **[4]** for a Whitney stratification ($WH_*(X)$ and $WH^*(X)$).

REFERENCES

[1] K. Bekka, *(C)-régularité et trivialité topologique*, Singularity theory and its applications, Warwick 1989, Part I, Lecture Notes in Math. 1462 (Springer, Berlin 1991), 42-62.

[2] K. Bekka, *Continuous vector fields and Thom's isotopy theorem*, preprint, Liverpool University, 1991.

[3] C. G. Gibson, K. Wirthmüller, A. A. du Plessis, E. J. N. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Math.552, Springer Verlag, 1976.

[4] M. Goresky, *Whitney stratified chains and cochains*, Trans. A. M. S. 267 (1981), 175-196.

[5] H. Hamm, *On stratified Morse theory*, Topology 38 (1999), 427-438.

[6] R. Hardt, D. Sullivan, *Variation of the Green function on Riemann surfaces and Whitney's holomorphic stratification conjecture*, Publications de l'I.H.E.S. 68 (1988), 115-138.

[7] J. Mather, *Notes on topological stability*, Mimeographed notes, Harvard University, 1970.

[8] J. Mather, *Stratifications and mappings*, Conference on Dynamical Systems (ed. M. M. Peixoto), Academic Press, New York, 1973, 195-232.

[9] C. McCrory, *Poincaré duality in spaces with singularities*, Thesis, Brandeis University, 1972.

[10] C. McCrory, *Stratified general position*, Algebraic and Geometric Topology, Santa Barbara 1977, Lecture Notes in Math. 664, Springer-Verlag, Berlin and New York, 1978, 142-146.

[11] C. Murolo, A. du Plessis, D. J. A. Trotman, *Stratified transversality by isotopy*, preprint.

[12] C. Murolo, D. J. A. Trotman, *Horizontally-C^1 controlled stratified maps and Thom's first isotopy theorem*, preprint.

[13] C. Murolo, *Semi-différentiabilité, transversalité et homologie de stratifications régulières*, thesis, Université de Provence, 1997.

[14] A. du Plessis, *Continuous controlled vector fields*, preprint.

[15] A. du Plessis, C. T. C. Wall, *The Geometry of Topological Stability*, Oxford University Press, Oxford, 1995.

[16] M. Shiota, *Piecewise linearization of real analytic functions*, Publ. R. I. M. S. 20 (1984), 724-792.

[17] M. Shiota, *Geometry of Subanalytic and Semialgebraic Sets*, Birkhäuser, Boston, 1997.

[18] S. Simon, *Champs totalement radiaux sur une structure de Thom-Mather*, Ann. Inst. Fourier, Grenoble, 45 (1995), 1423-1447.

[19] R. Thom, *Ensembles et morphismes stratifiés*, Bull. A. M. S. 75 (1969), 240-284.

[20] H. Whitney, *Local properties of analytic varieties*, Differential and Combinatorial Topology (ed. S. Cairns), Princeton University Press, Princeton (1969), 205-244.

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