STRATIFIED TRANSVERSALITY VIA TIME-DEPENDENT VECTOR FIELDS

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Abstract. For X a (w)-regular or (c)-regular stratification, hence for any Whitney stratification and, via regular embedding, for any abstract stratified set, we use time-dependent vector fields to prove an extension theorem for diffeomorphisms near the indentity defined on strata of a given dimension. Then we show that after isotopy a stratified map $h : Z \to X$ can be made transverse to a fixed stratified map $g : Y \to X$.

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1. Introduction. A substratified object of a stratified space $X = (A, \Sigma)$ is a stratified space $W = (W, \Sigma_W)$ such that $W \subseteq A$ and each stratum of Σ_W is contained in one and only one stratum of the stratification Σ of X. We consider the problem of putting a substratified object W, or a stratified map $h : Z \to X$, in transverse position with respect to a stratified map $g : Y \to X$.

This problem was previously considered by Clint McCrory who solved it for stratified polyhedra **[13]** giving a result essential to the foundation of intersection homology. The problem was treated in the context of Whitney stratifications ((b)-regular) by Mark Goresky, first in his thesis **[7]** and later in **[8]** (*Transversality Lemma 5.3*) with a completely revised formulation.

Goresky's *Transversality Lemma* is for substratified objects W of X satisfying the *π-fibre condition* and a stratified map $g : Y \to X$ which is controlled with respect to two systems of control data. The π-fibre condition for W and the control condition for $g : Y \to X$ are conditions implying that W is locally, near each point x of A, a union of stratified fibres of a projection $\pi_S : T_S \to S$ (S stratum of X containing x) and a similar geometric property for the fibres of g. These strong conditions were essential to Goresky's inductive sketch proof in order to preserve the transversality with respect to g of a deformation W' of W.

The utility of the transversality theorem of Goresky was proven already in various significant applications : first it was the key result in proving the main theorems 3.4, 4.7 and 6.2 of **[8]** on representing homology and cohomology of a Whitney stratification X by Whitney cycles and cocycles, then the first author of the present paper used it to define a geometric sum operation in the Goresky sets $WH^k(X)$ and to give a geometric construction of the Steenrod cohomology operations **[15, 16]**. On the other hand, the hypotheses of π-fibre on W, and control on f, prevent more general applications.

In this article we prove a stratified transversality theorem for two stratified maps $h : Z \to X$ and $g : Y \to X$ between abstract stratified sets or (w)-regular stratifications, without assuming any control condition. Abstract stratified sets form a class containing Whitney and (c)-regular stratifications; but not all (w)-regular stratifications.

The authors gave recently a different proof of this result **[18]**. The proof presented here gives more information and is based on a theorem of extension of stratified homeomorphisms and is in reality the original generalisation (see [**17**]).

The content of the paper is as follows.

In §2 we recall the definitions of stratified sets of Thom-Mather [**5, 11, 12**], (c)-regular stratifications, stratified and controlled maps, and stratified time-dependent vector fields.

In §3 we prove a stratified extension theorem (Theorem 3.1) which allows one to extend *globally* on the whole of A diffeomorphisms given on the k-dimensional part $S = A_k - A_{k-1}$ of A which extend continuously the identity on A_{k-1}. We initially state and prove it for the (c)-regular stratifications of K. Bekka [**1**], and then generalize to a more general stratified context (theorem 3.2). Recall that Whitney (b)-regular stratifications are (c)-regular [**1**].

In order to prove theorems 3.1 and 3.2, we use certain time-dependent vector field techniques introduced by Mather [**10**] (to show that infinitesimal stability implies stability) in proving that *"every diffeomorphism $f : S \to S$ in a small enough neighborhood of 1_S in $Diff(S, S)$ is the flow at time $t = 1$ of a time-dependent vector field"* (such a flow *is not* in general a one parameter group [**4**]).

Theorem 3.2 states that the stratified extension theorem holds for abstract stratified sets [**11, 12**], and also for (w)-regular stratifications [**26**] (thus for Lipschitz stratifications [**21**]).

In §4 we give the transversality theorem (theorem 4.8) : for every pair of stratified maps $g : Y \to X$ and $h : Z \to X$ there exists a *deformation by isotopy* h' of h in X which is transverse to g. Theorem 4.8 holds for abstract stratified sets and for (w)-regular stratifications.

We conclude the article by applying theorem 4.8 to deform, via a stratified isotopy $\Phi_1 : X \to X$, a substratified object W of X to a substratified object $W' = \Phi_1(W)$ transverse to a given map $g : Y \to X$. We obtain thus Corollary 4.12 which generalizes Goresky's Transversality Lemma to abstract stratified sets which are not necessarily π-fibre and to stratified maps which are not necessarily controlled.

2. Stratified sets and maps. A C^k *stratification* ($1 \leq k \leq \infty$) of a topological space A is a locally finite partition Σ of A into C^k connected manifolds (the *strata* of Σ) satisfying the *frontier condition* [**11**] : if X and Y are distinct strata such that $X \cap \overline{Y} \neq \emptyset$, then $X \subseteq \overline{Y}$ and we write $X < Y$. The pair $X = (A, \Sigma)$ is called a C^k *stratified space* with *stratification* Σ, and the union of the strata of dimension $\leq k$, the *k-skeleton* A_k, induces a C^k stratified space $X_k = (A_k, \Sigma_{|A_k})$.

Extra conditions may be imposed on Σ, such as to be an *abstract stratified set* of Thom-Mather [**5, 11, 12**] or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney [**11, 12, 27**], or (c) of Bekka [**1**] or, when A is a subset of a C^2 manifold, to satisfy condition (w) of Kuo-Verdier [**26**] or (L) of Mostowski [**21**].

Definition 2.1. Let X be a stratified space. A family $F = \{(\pi_X, \rho_X, T_X)\}_{X \in \Sigma}$ is called *a system of control data* for X if for each stratum X of X we have :

1) T_X is a neighbourhood of X in A (called a *tubular neighbourhood of* X);
2) $\pi_X : T_X \to X$ is a continuous retraction of T_X onto X (called *projection on* X);
3) $\rho_X : T_X \to [0, \infty[$ is continuous and $X = \rho_X^{-1}(0)$ (ρ_X is the *distance function from* X)

and for every pair $X < Y$ of strata, the restrictions $\pi_{XY} = \pi_{X|T_{XY}}$ and $\rho_{XY} = \rho_{X|T_{XY}}$, to $T_{XY} = T_X \cap Y$, satisfy :

5) $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0, \infty[$ is a smooth submersion (thus $\dim X < \dim Y$);
6) for every stratum Z such that $Z > Y > X$ and for every $z \in T_{YZ} \cap T_{XZ}$ one has :
i) $\pi_{XY}\pi_{YZ}(z) = \pi_{XZ}(z)$ (the *π-control condition*)

ii) $\rho_{XY}\pi_{YZ}(z) = \rho_{XZ}(z)$ (the *ρ-control condition*).

In what follows let $T_X(\epsilon) = \rho_X^{-1}([0,\epsilon[)\,, \forall\, \epsilon \geq 0$, and assume $T_X = T_X(1)$.

If A is Hausdorff, locally compact and admits a countable basis for its topology, the pair $(\mathcal{X}, \mathcal{F})$ is called an *abstract stratified set* (since one works with a unique system of control data of $\mathcal{X}$, in what follows we omit $\mathcal{F}$).

If $\mathcal{X}$ is an abstract stratified set, the tubular neighbourhoods $\{T_X\}_{X\in\Sigma}$ may be chosen such that: "$T_{XY} \neq \emptyset$ if and only if $X < Y$ or $X > Y$ or $X = Y$" (see [**12**]).

A *stratified map* $f : \mathcal{X} \to \mathcal{X}'$ between stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (B, \Sigma')$ is a continuous map $f : A \to B$ which sends each stratum X of $\mathcal{X}$ into a unique stratum X' of $\mathcal{X}'$, such that the restriction $f_X : X \to X'$ is smooth. Call f a *stratified homeomorphism* if f is a global homeomorphism and each f_X is a diffeomorphism.

Given $f : \mathcal{X} \to \mathcal{X}'$ and two systems of control data $\mathcal{F} = \{(T_X, \pi_X, \rho_X)\}_{X\in\Sigma}$ and $\mathcal{F}' = \{(T_{X'}, \pi_{X'}, \rho_{X'})\}_{X'\in\Sigma'}$, f is called *controlled (with respect to $\mathcal{F}$ and $\mathcal{F}'$)* if $\forall\, X < Y$, $\exists\, \epsilon > 0$ such that for all $y \in T_{XY}(\epsilon) = T_X(\epsilon) \cap Y$ we have :

$$\begin{cases} \pi_{X'Y'}f_Y(y) = f_X\pi_{XY}(y) & \text{(the } \pi\text{-}\textit{control condition} \text{ for } f) \\ \rho_{X'Y'}f_Y(y) = \rho_{XY}(y) & \text{(the } \rho\text{-}\textit{control condition} \text{ for } f). \end{cases}$$

A stratified vector field on $\mathcal{X}$ is a family $\zeta = \{\zeta_X\}_{X\in\Sigma}$ of vector fields, with every ζ_X a smooth vector field on the stratum X. Similarly, a stratified vector field $\zeta = \{\zeta_X\}_{X\in\Sigma}$ is called (π,ρ)*-controlled (with respect to $\mathcal{F}$)* if the following two *control conditions* hold:

$$\begin{cases} \pi_{XY*}(\zeta_Y(y)) = \zeta_X(\pi_{XY}(y)) & \text{(the } \pi\text{-}\textit{control condition} \text{ for } \zeta) \\ \rho_{XY*}(\zeta_Y(y)) = 0 & \text{(the } \rho\text{-}\textit{control condition} \text{ for } \zeta). \end{cases}$$

The notion of system of control data of $\mathcal{X}$, introduced by Mather in [**12**], is the fundamental tool allowing one to obtain good extensions of vector fields.

PROPOSITION 2.2 ([**5, 12**]). *Let $\mathcal{X}$ be a C^2 abstract stratified set, X a stratum of $\mathcal{X}$. For each C^1 vector field ζ_X on X there exists a stratified (π,ρ)-controlled lifting $\zeta_{T_X} = \{\zeta_Y\}_{Y\geq X}$ defined on a tubular neighbourhood T_X of X. Moreover, if ζ_X admits a global flow $\{\phi_t : X \to X\}_{t\in\mathbb{R}}$, then ζ_{T_X} admits a global flow $\{\phi_{T_X,t} : T_X \to T_X\}_{t\in\mathbb{R}}$, such that each $\phi_{T_X,t}$ is a stratified, (π,ρ)-controlled homeomorphism.*

DEFINITION 2.3. (K. Bekka 1990). A C^k stratified space (A, Σ) with $A \subseteq \mathbb{R}^n$, is called (c)*-regular* if, for every stratum $X \in \Sigma$, there exists an open neighbourhood U_X of X in $\mathbb{R}^n$ and a C^1 function $\rho_X : U_X \to [0, \infty[$, such that $\rho_X^{-1}(0) = X$, and such that its stratified restriction to the *star* of X :

$$\rho_X \ : \ Star(X) \cap U_X \to [0, \infty[\qquad \text{is a Thom map,}$$

where $Star(X) = \cup_{Y\in\Sigma\,,\, Y\geq X} Y$ and the stratification on $Star(X) \cap U_X$ is induced by Σ.

In substance, the (c)-regularity of Bekka means that for every pair of adjacent strata $X < Y$, the tangent spaces at $y \in Y$ to the level hypersurfaces $\rho_X^{-1}(\epsilon)$ (where $\epsilon = \rho_X(y)$) have limits which contain T_xX when $y \to x \in X$.

REMARK 2.4. A Bekka (c)-regular stratified space $\mathcal{X} = (A, \Sigma)$ admits a system of control data $\{(\pi_X, \rho_X, T_X)\}_{X\in\Sigma}$ in which for each stratum $X \in \Sigma$, $T_X = U_X \cap A$, and π_X, ρ_X are restrictions of C^∞ maps defined on U_X [**1**]. Thus (c)-regular stratifications admit a structure of abstract stratified set and so proposition 2.2. holds for them. We underline moreover that in this case, for each vector field ζ_X on a stratum X of A, the stratified

(π, ρ)-controlled lifting $\zeta_{T_X} = \{\zeta_Y\}_{Y \geq X}$ defined on a tubular neighbourhood T_X of X may be chosen *continuous* **[2]** (this gives a more regular lifted flow $\{\phi_{T_X,t} : T_X \to T_X\}_{t\in\mathbb{R}}$).

Stratified time-dependent vector fields. Let S be a C^∞ manifold, and let $I = [0,1]$ be the closed unit interval. With notation as in **[10]** (page 286) and **[23]** (page 61), consider the projection $\pi_S : S \times I \to S$ onto the first factor.

Recall that the fibre bundle pullback $(\pi_S^*TS, \Pi_{S\times I}, S \times I)$ with base space $S \times I$ of the tangent bundle TS has for total space :

$$\pi_S^*TS = \big\{\big((x,t),(y,v)\big) \in (S \times I) \times TS \;\mid\; x = y\big\}$$

and so identifying each element $((x,t),(x,v)) \in \pi_S^*TS$ with (x,v,t), one can rewrite as :

$$\pi_S^*TS \;\; = \cup_{(x,t)\in S\times I}\{(x,t)\} \times T_xS \;\; = \;\; TS \times I\,.$$

The fibre of the projection $\Pi_{S\times I} : \pi_S^*TS \to S \times I$ at a point (x,t) is T_xS and so the space of sections $\Gamma^\infty(\pi_S^*TS)$ is exactly the space of C^∞ level preserving vector fields $\zeta = \{\zeta_t\}_{t\in I}$ defined on $S \times I$.

Definition 2.5. A level preserving vector field $\zeta \in \Gamma^\infty(\pi_S^*TS)$ is called a *time-dependent vector field on the manifold* S. We will also denote ζ by $\{\zeta_t\}_{t\in I}$, thinking of it as a smooth family of vector fields on S.

Consider a time-dependent vector field $\zeta = \{\zeta_t\}_{t\in I}$ and the differential equation on S:

$$E(\zeta) \quad : \quad \begin{cases} \frac{\partial}{\partial t}\beta(x,t) = \zeta_t(\beta(x,t)) \\ \beta(x,0) = x. \end{cases}$$

A *global flow of* $\zeta = \{\zeta_t\}_t$ is a smooth map $\phi : S \times I \to S$ which is a solution of the equation $E(\zeta)$. We write also $\phi = \{\phi_t : S \to S\}_t$.

Mather showed (Lemma 2, page 289 of **[10]**, or Lemma 3.7.5 in **[23]**) that every time-dependent vector field $\zeta = \{\zeta_t\}_t$ in a sufficiently small neighbourhood O_S in $\Gamma^\infty(\pi_S^*TS)$ of the zero (time-dependent) vector field $\mathbf{0}$ admits such a global flow.

Let $X = (A, \Sigma)$ be a C^2 abstract stratified set equipped with a system of control data $F = \{(T_X, \pi_X, \rho_X)\}_{X\in\Sigma}$.

Fix a stratum X of A and a time-dependent vector field $\zeta_X = \{\zeta_{X\,t}\}_t$. For all $t \in I$, $\zeta_{X\,t}$ is a vector field of X, so (by proposition 2.2) we can consider a (π, ρ)-controlled lifting $\zeta_{T_X\,t}$ on the tubular neighbourhood T_X of X. We thus determine a time-dependent vector field $\zeta_{T_X} = \{\zeta_{T_X\,t}\}_t$ on T_X. Integrability and continuity of the controlled lifted flow of the vector field defined on a stratum **[5, 11]** still follow when we consider on X and T_X time-dependent vector fields instead of the usual vector fields. In particular, with notations as above, we find :

Proposition 2.6. *Let X be a stratum of a C^2 abstract stratified set (resp. (c)-regular stratification) $X = (A, \Sigma)$. Let $\zeta_{T_X} = \{\zeta_{T_X\,t}\}_t$ be a time-dependent vector field on T_X obtained as a controlled (resp. continuous controlled) lifting of a time-dependent vector field $\zeta_X = \{\zeta_{X\,t}\}_t$ defined on X. Then the flow $\phi_{T_X} = \{\phi_{T_X\,t}\}_t$ of ζ_{T_X} satisfies the (π, ρ)-control conditions and is a continuous extension of the flow $\phi_X = \{\phi_{X\,t}\}_t$ of ζ_X.*

Proof. Thanks to the ρ-control condition on each $\zeta_{X\,t}$ **[5, 11, 12]**, for each $y \in T_X$ the function $\rho_X \circ \phi_{T_X}(y,t)$ is constant as t varies, and the trajectory $\phi_{T_X}(y,t)$ stays in the level hypersurface $\rho_X^{-1}\big(\rho_X(y)\big)$. The proof is then formally the same as in the classical case **[5, 11, 12]**) of a lifted flow of a vector field which does not depend on time. $\square$

3. Extension of stratified homeomorphisms.

The stratified extension theorem proved below is based on techniques of Mather used in proving that infinitesimal stability of a map $f \in C^\infty(S,S)$ implies stability [**10**].

We first give the extension theorem for (c)-regular stratifications [**1**] of a closed subset of a manifold M.

STRATIFIED EXTENSION THEOREM 3.1. *Let $X = (A, \Sigma)$ be a C^∞ (c)-regular stratification of a closed subset A of a C^∞ manifold M, and let S denote the union of the strata of dimension k of X.*

There exists a neighbourhood B of 1_S in $Diff(S,S)$ such that each $f \in B$ extends to a stratified homeomorphism $\tilde{f} : A \to A$ such that $\tilde{f}_{|A_{k-1}} = 1_{A_{k-1}}$. Moreover there is a stratified isotopy $\Phi : A \times I \to A$ such that $\tilde{f} = \Phi_1$, the section at time $t = 1$ of Φ.

Proof. We adopt the notation of [**10**] and [**23**, chapter III]. Spaces of maps will be given the Whitney C^∞ topology.

Step 1 : Construction, for f near 1_S, of an isotopy γ_f between 1_S and f which is the flow of a time-dependent vector field on S.

Consider a *"family of geodesics"* of the smooth manifold S (see [**10**] or [**23** Proposition 3.3.1]), i.e. a smooth map

$$\gamma : N \times I \longrightarrow S \quad \text{such that} \quad \begin{cases} \gamma(x,y,0) = x & \forall\,(x,y) \in N \\ \gamma(x,x,t) = x & \forall\, x \in S \ \text{ and } \ \forall\, t \in I \\ \gamma(x,y,1) = y & \forall\,(x,y) \in N \end{cases}$$

where N is a neighbourhood of the diagonal in $S \times S$.

As in corollary 3.4.5 of [**23**], with $W = \{g \in C^\infty(S,S) \mid (x, g(x)) \in N\}$ an open neighbourhood of 1_S in $C^\infty(S,S)$, we may suppose the continuity of the map

$$G \ : \ W \longrightarrow C^\infty(S \times I, S \times I)\,, \quad \text{defined for } f \in W \text{ by} \quad G(f)(x,t) = \big(\gamma\big(x, f(x), t\big), t\big).$$

Writing $G(f)_t(x) = G(f)(x,t)$, we have : $G(f)_0 = 1_S$ and $G(f)_1 = f$.

For fixed f, define $\phi : S \times I \to I$ by $\phi(x,t) = \phi_t(x) = \gamma(x, f(x), t)$, i.e. $\phi_t = G(f)_t$. So $\phi_0 = 1_S$ and $\phi_1 = f$.

We will prove that if U is a sufficiently small neighbourhood of 1_S in $Diff(S,S)$,then for each $f \in U$ the path $\gamma_f(t) = \gamma(x, f(x), t)$ is contained in U and defines an isotopy between 1_S and f which is the flow of a time-dependent vector field.

Let U be an open neighbourhood of 1_S in $Diff(S,S)$.

After possibly shrinking W, we can assume $W \subseteq U$, so that $W \subseteq Diff(S,S)$.

By Lemma 3.4.9 of [**23**], there exists an open neighbourhood V of the identity $1_{S\times I}$ in $C^\infty(S \times I, S \times I)$, such that each $F \in V$ is level-preserving, i.e. can be written $F(x,t) = (f_t(x), t)$ and moreover for each t, $f_t \in U$. It will be convenient to write $F = \{f_t : S \to S\}_{t \in I}$.

With the notations of [**23**], we set $V = \big\{F \in C^\infty_{lp}(S \times I, S \times I) \ \mid \ \lambda_F(I) \subseteq U \ \big\}$ where *"lp"* means *"level-preserving"*, and $\lambda_F(t) = f_t$ for $t \in I$.

When $F \in V$, we see that F is a level-preserving *diffeomorphism* of $S \times I$, because $U \subseteq Diff(S,S)$. Thus V is an open neighbourhood of $1_{S\times I}$ contained in $Diff_{lp}(S \times I, S \times I)$.

Because $G : W \to C^\infty(S \times I, S \times I)$ is continuous, $W' = G^{-1}(V)$ is an open neighbourhood of 1_S contained in U and

$$G(W') \ \subseteq \ V \ \subseteq \ Diff_{lp}(S \times I, S \times I)\,.$$

For each $f \in W'$, $G(f)$ is a level-preserving diffeomorphism of $S \times I$, i.e. $G(f) = \{G(f)_t\}_{t \in I}$, where each diffeomorphism $G(f)_t$ lies in U.

In particular, if $f \in W'$ the map $\phi_t = G(f)_t : S \to S$ is a diffeomorphism, the path $\gamma_f(t) = \phi_t$ defines an isotopy of S, and the image of γ_f is contained in U.

Now, given $f \in W'$ and $t \in I$, consider the vector field :

$$\xi_t(x) \;=\; \frac{\partial}{\partial \tau}\Big|_{\tau=0} \phi_{t+\tau}(x) \;=\; \frac{\partial}{\partial \tau}\Big|_{\tau=0} \gamma\big(x, f(x), t+\tau\big) \,.$$

As $\xi_t(x)$ is tangent to S at $\phi_t(x) = \gamma(x, f(x), t)$, and ϕ_t is a diffeomorphism for each $t \in I$, the composed map

$$\begin{array}{rcccccc} \zeta_t \;=\; \xi_t \circ \phi_t^{-1} & : & S & \to & S & \to & TS \\ & & x & \mapsto & \phi_t^{-1}(x) & \mapsto & \xi_t(\phi_t^{-1}(x)) \end{array}$$

defines a vector field on S.

Also

$$\zeta_t(\phi_t(x)) = \xi_t(x) = \frac{\partial}{\partial \tau}\Big|_{\tau=0} \phi_{t+\tau}(x) = \frac{\partial}{\partial t}\phi(x,t),$$

so that $\phi : S \times I \to S$ is the flow of the time-dependent vector field $\zeta = \{\zeta_t\}_{t \times I} \in \Gamma^\infty(\pi_S^* TS)$. Recall that $\phi(x,0) = \phi_0(x) = 1_S(x) = x$ and

$$\phi_1(x) = \phi(x,1) = \gamma(x, f(x), 1) = f(x), \quad \text{ i.e. } \quad \phi_1 = f\,,$$

by the properties of the family of geodesics γ.

Precision in the choice of U. To find the neighbourhood B claimed in the statement of the theorem it is useful to choose carefully U. More regularity of $f \in U$ on $\overline{S} - S$ will give more regularity of the extension $f_{T_S} : T_S \to T_S$ of f in “step 2”.

Let us consider for U the open neighbourhood of 1_S in $C^\infty(S,S)$ with the C^l-topology

$$U = B^l_\delta(1_S) = \big\{ g \in Diff(S,S) \quad \big| \quad d_l(j^l g(x), j^l 1_S(x)) < \delta(x) \;,\; \forall x \in S \;\big\}$$

and hence with the C^∞-topology (notation of [**6**]) where $\delta : S \to]0,1[$ is a function having a smooth extension by zero on $\partial S = \overline{S} - S = \bigcup_{R<S} R$ with all its derivatives.

By choice of δ, each $f \in U = B^l_\delta(1_S)$ tends to 1_S on ∂S and since $l \geq 1$, such an extension has a differential which is the identity on ∂S : i.e. for every sequence $\{x_n\}$ in S such that $\lim x_n = x \in \partial S$ and $\lim_n T_{x_n} S = \sigma$ we have that $\lim_n f_{*x_n} = id_\sigma$.

Recall that the time-dependent vector field $\zeta = \{\zeta_t\}_t \in \Gamma^\infty(\pi_S^* TS)$ defined in step 1 depends on a choice of $f \in W'$ (f was in the definition of $\phi_t = G(f)_t$); so we write $\zeta(f) = \{\zeta(f)_t\}_t$. This gives a continuous map :

$$\tfrac{\partial G}{\partial t} \;=\; G' \,:\, W' \longrightarrow \Gamma^\infty(\pi_S^* TS) \quad , \quad G'(f) \;=\; \zeta(f) \;:\; S \times I \to \pi_S^* TS = TS \times I$$

defined on the neighbourhood W' of 1_S in $U \subseteq Diff(S,S)$ [**10, 23**].

Mather showed (Lemma 2 on page 289 of [**10**], cf. Lemma 3.7.5 of [**23**]) that there exists a neighbourhood O_S of the zero vector field $\mathbf{0}$ in $\Gamma^\infty(\pi_S^* TS)$ such that every time-dependent vector field $\zeta \in O_S$ admits a flow $\beta = \beta(\zeta)$ defined for all time $t \in I$:

$$\beta = \beta(\zeta) : S \times I \to S \qquad \text{such that} \qquad \begin{cases} \beta(\zeta)_0 = 1_S \\ \text{ and} \\ \beta(\zeta)_t \in Diff(S,S) \;,\; \forall\, t \in I \end{cases}$$

and such that the map

$$\theta_S : \Gamma^\infty(\pi_S^* TS) \to C^\infty(S \times I, S) \quad , \quad \theta_S(\zeta) = \beta(\zeta) \ : \ S \times I \longrightarrow S$$

is continuous.

Choose a neighbourhood $B^l_\epsilon(0)$ of the zero vector field $0 \in \Gamma^\infty(S)$ and a neighbourhood $\tilde{B}^l_\eta(\mathbf{0})$ of $\mathbf{0} \in \Gamma^\infty(\pi_S^* TS)$ such that for each $\zeta = \{\zeta_t\}_{t\in I} \in \tilde{B}^l_\eta(\mathbf{0})$ we have $\zeta_t \in B^l_\epsilon(0)$ for all $t \in I$ (these exist by Lemma 3.4.9 of **[23]**), where ϵ, $\eta : S \to (0,1)$ are smooth functions which tend, with all their derivatives, to zero on the frontier ∂S of S.

Set $B = {G'}^{-1}\big(O_S \cap \tilde{B}^l_\eta(\mathbf{0})\big)$. As G' is continuous and $G'(1_S) = \mathbf{0}$, B is an open neighbourhood of 1_S in $Diff(S,S)$ such that $B \subseteq W' \subseteq B^l_\delta(1_S) = U$ and the following properties *i), . . . , iv)* hold for each $f \in B$:

i) Because $f \in B \subseteq B^l_\delta(1_S)$, f extends continuously by 1_S on ∂S and this extension has a differential which extends continuously by the identity on ∂S, i.e. for any sequence $\{x_n\}$ in S such that $\lim x_n = x \in \partial S$ and $\lim_n T_{x_n} S = \tau$ we have that $\lim_n f_{*x_n} = id_\tau$.

ii) Since $f \in {G'}^{-1}\big(\tilde{B}^l_\eta(\mathbf{0})\big)$, the time-dependent vector field $G'(f) = \zeta(f) = \{\zeta(f)_t\}_t$ is such that each level vector field $\zeta(f)_t : S \to TS$, together with its derivatives, extends continuously by zero on ∂S, because $\zeta(f)_t \in B^l_\epsilon(0)$.

iii) As $G'(f) \in O_S$, the time-dependent vector field $G'(f) = \zeta(f) = \{\zeta(f)_t\}_{t\in I}$ admits a flow $\beta(\zeta(f)) : S \times I \to S$.

iv) Since $f \in W'$, writing $\beta(\zeta(f)) = \{\beta(\zeta(f))_t\}_t$ and $\phi_t = \beta(\zeta(f))_t$, we have that $\phi_0 = 1_S$ et $\phi_1 = f$ (by construction, see also Lemma 1 on page 289 of **[10]**).

Step 2 : Constructing an extension f_{T_S} of f, on strata of dimension $\geq k$ in a tubular neighbourhood $T_S = \cup_{\dim X = k} T_X$ of S, such that f_{T_S} tends to the identity on ∂S.

For each k-stratum X of A, i.e. each connected component X of S, choose a tubular neighbourhood $T_X = T_X(1)$ of X in A; these neighbourhoods $\{T_X\}_X$ may be chosen pairwise disjoint **[12]**. Denote by $f_X : X \to X$ the restriction of f to X.

Let $f \in B \subseteq Diff(S,S)$. Then f preserves each connected component of S, and we can write $f \ = \cup_{\dim X = k} f_X$ with $f_X \in Diff(X,X)$ for each k-stratum X of A.

To extend $f : S \to S$ to a stratified homeomorphism $f_{T_S} : T_S \to T_S$, which is a diffeomorphism on each stratum of $T_S = \cup_{\dim X = k} T_X$, we define an extension $f_{T_X} : T_X \to T_X$ of each f_X and take the (disjoint) union $f_{T_S} = \cup_X f_{T_X}$.

Remark that each level vector field $\zeta(f)_t$ of S is a disjoint union $\zeta(f)_t = \cup_X \zeta(f)_{t\,X}$.

Let $\zeta(f)_{T_S} = \{\zeta(f)_{t,T_S}\}_{t\in I}$ be the stratified time-dependent vector field obtained by controlled lifting to $T_S = \cup_X T_X$ of the level vector fields $\zeta(f)_t$ on S.

As the stratification $X = (A, \Sigma)$ is (c)-regular, we can assume that each $\zeta(f)_{t,T_S}$ is a *continuous* lifting of $\zeta(f)_t$ on S **[2, 22]**.

Since the time-dependent vector field $\zeta(f)_{T_S}$ is a controlled lifting of $\zeta(f)$ which admits a flow $\phi : S \times I \to S$, then, by Proposition 2.4, $\zeta(f)_{T_S}$ admits a flow defined for all $t \in I$, namely $\phi_{T_S} : T_S \times I \to T_S$, which is a continuous extension of ϕ.

Then for each $f \in B$, the map at time $t = 1$, $f_{T_S} = \big[\phi_{T_S}\big]_1 : T_S \to T_S$ is a continuous extension of $f : S \to S$. Moreover, as the map $\Psi : f \mapsto f^{-1}$ is continuous **[10, 23]**, replacing B by $B \cup \Psi(B)$ after shrinking B, we can suppose "$f \in B \Rightarrow f^{-1} \in B$". One finds that $(f_{T_S})^{-1} = \beta(\zeta(f^{-1})_{T_S})_1$, thus f_{T_S} is a stratified homeomorphism extending f.

Finally, since $f \in B$, $\zeta(f)_t \in B^l_\epsilon(0)$ for all $t \in I$, and so the continuous extension $\zeta(f)_{t\,T_S}$ also extends continuously by the zero vector field $\mathbf{0}$ on ∂S. We have :

$$\lim_{z \to x' \in \partial S} f_{T_S}(z) = x'.$$

Step 3 : Extension of the restriction $f_{T_S(\frac{1}{2})}$ to the whole of A. So as to obtain an extension of the diffeomorphism $f \in B$ to A, we give a "deformation" of $f_{T_S(1)}$ on $T_S(1) - \overline{T_S(\frac{1}{2})}$ which tends to the identity on $A - T_S(1)$ and to $f_{T_S(1)}$ on $T_S(\frac{1}{2})$.

Since $f_{T_S(1)}$ is defined by the family of vector fields $\zeta(f)_{T_S} = \{\zeta(f)_{t\,T_S}\}_{t \in I}$, we can modify the modulus of each lifted controlled vector field $\zeta(f)_{t,T_S}$ as follows. Write $\zeta(f)_{t\,T_S} = \cup_{\dim X = k} \zeta(f)_{t\,T_X}$, and choose a smooth real function $g : \mathbb{R} \to [0,1]$ such that $g(s) = 1$ if $s \leq \frac{1}{2}$ and $g(s) = 0$ if $s \geq 1$. We define then the time-dependent vector field

$$\widetilde{\zeta(f)_{t}}_{T_X}(z) = g(\rho_X(z)) \cdot \zeta(f)_{t\,T_X}(z),$$

so that

$$\widetilde{\zeta(f)}_{T_X} = \begin{cases} \zeta(f)_{T_X} & \text{if} \quad \rho_X(z) \leq \frac{1}{2}, \\ 0 & \text{if} \quad \rho_X(z) \geq 1 \end{cases}$$

and it is controlled since every $\zeta(f)_{T_X}$ is controlled in $T_X(\frac{1}{2}) \subset T_X(1) = T_X$.

The flows $\tilde{\phi}_{T_X}$ of $\widetilde{\zeta(f)}_{T_X}$ and ϕ_{T_X} of $\zeta(f)_{T_X}$ satisfy $\tilde{\phi}_{T_X}(z,t) = \phi_{T_X}\big(z\,,\, g(\rho_X(z)) \cdot t)\big)$ and since ϕ_{T_X} is defined for all $t \in I$, this also holds for $\tilde{\phi}_{T_X}$.

Finally, because the neighbourhoods $\{T_X\}_{\dim X = k}$ are pairwise disjoint, we can define a global time-dependent vector field on A :

$$\zeta(f)_A(z) = \begin{cases} \widetilde{\zeta(f)}_{T_X}(z) & \text{if } z \in \quad \cup_{\dim X=k} T_X(1) \\ 0 & \text{if } z \in A - \cup_{\dim X=k} T_X(1)\,. \end{cases}$$

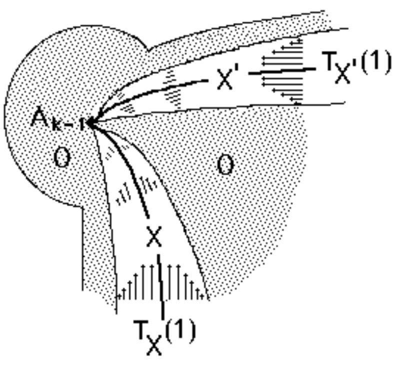

figure 1

The time-dependent vector field $\zeta(f)_A$ clearly has a global flow, namely $\beta(\zeta(f)_A)$: $A \times I \to A$, which extends continuously $\tilde{\phi}_{T_X}$ by the identity outside $T_X(1)$.

Finally, for each $f \in B$, the claimed extension $\tilde{f} : A \to A$ is found by taking the map at time $t = 1$, $\tilde{f} = \big[\beta(\zeta(f)_A)\big]_1 : A \to A$. In fact, since for every $t \in I$, $\big[\beta(\zeta(f)_A)\big]_t : A \to A$ is a stratified homeomorphism, $\tilde{f}$ is a stratified homeomorphism (see also proposition 2.4); since $\widetilde{\zeta(f)}_A = 0$ on A_{k-1} then $\tilde{f}_{|A_{k-1}} = 1_{A_{k-1}}$; and since $\tilde{f} = \big[\beta(\zeta(f)_A)\big]_1$ and $f = \big[\beta(\zeta(f))\big]_1$ (see step 1) with $\beta(\zeta(f)_A)$ extending $\beta(\zeta(f))$, then $\tilde{f} : A \to A$ extends $f : S \to S$. $\square$

THEOREM 3.2. *The statement of theorem 3.1 is valid if X is a C^∞ abstract stratified set or a C^∞ (w)-regular stratification of a closed subset of a C^∞ manifold.*

Proof. If X is an abstract stratified set **[11]** we can find a (b)-regular embedding of X in some $\mathbb{R}^m$ (proved to exist in **[7, 19, 24]**; see also **[20]** for a (w)-regular subanalytic, hence (b)-regular embedding) and apply Theorem 3.1. using that (b) implies (c) **[1]**.

If X is a (w)-regular stratification, it may not be possible to put an abstract stratified structure on A, and (c)-regular embeddings may not exist (for example if A is the closure in $\mathbb{R}^2$ of $\{(x,y) \mid y = sin(1/x), x \neq 0\}$; this is not triangulable so is not even homeomorphic to an abstract stratified set).

However, when X is (w)-regular a vector field $\zeta(f)_X$ on a stratum X admits a stratified *rugose* lifting $\zeta(f)_{T_X}$ whose flow is (rugose and hence) continuous **[3, 26]**. The only change to the proof of Theorem 3.1 is to specify the $\{T_X\}$ as pairwise disjoint neighbourhoods of the strata of dimension k (instead of tubular neighbourhoods associated to a system of control data), and for each X let ρ_X be a smooth distance function.

The vector field $\cup_X \widetilde{\zeta(f)}_{T_X}$, starting from $\cup_X \zeta(f)_{T_X}$, and the final extension $\zeta(f)_A$ on A, are obtained as in Theorem 3.1. $\square$

Remark 3.3. In the stratified extension theorems 3.1 and 3.2 if A is a closed stratified subset of a manifold M it is easy to see that an extension exists on the whole of M, by treating M as a stratified set of which A is a union of strata (add the components of $M - A$ as strata of codimension 0 in M).

4. Stratified Transversality of maps and of substratified objects.

In this section we use the stratified extension theorem 3.1 to prove a stratified transversality theorem in which we deform a stratified map $h : Z \to X$ to make it transverse to a fixed stratified map $g : Y \to X$. A different proof of the same theorem has been recently obtained by the authors in **[18]**.

Definition 4.1. Let $X = (A, \Sigma)$ be a stratified space. A *stratified isotopy* of X (or of A) is a stratified map $\Phi : A \times I \to A$, $\{\Phi_t : A \to A\}_{t \in I}$ (where $I = [0,1]$), such that for every $t \in I$, the map $\Phi_t : A \to A, \Phi_t(x) = \Phi(x,t)$ is a stratified homeomorphism. Clearly, if $\{\Phi_t\}_t$ and $\{\Psi_t\}_t$ are stratified isotopies of X then so is the composite map $\{\Psi_t \circ \Phi_t\}_t$.

Definition 4.2. Let $h, h' : Y \to X$ be two stratified maps. We say that h' *is a deformation by isotopy of h in X* if there exists a stratified isotopy $\Phi : X \times I \to X$ such that $\Phi_0 = 1_X$ and $h' = \Phi_1 \circ h$, i.e. h' is the deformation via Φ and at time $t = 1$ of h. We write then $h \overset{\Phi_t}{\equiv} h'$.

Remark 4.3. *The relation of "deformation by isotopy in X" is an equivalence relation on the set of stratified maps $h : Y \to X$.*

Remark 4.4. *If $\Phi : A \times I \to A$ is a stratified isotopy of $X = (A, \Sigma)$ and $h : Y \to X$ is a stratified map, for each $t \in I$, $h_t = \Phi_t \circ h$ is a deformation by isotopy of h in X.*

Definition 4.5. Two stratified maps $h, g : Y \to X$ are said to be *transverse in the k-skeleton X_k (or A_k) of X* when for each stratum S of X_k their restrictions to S are transverse in S. More precisely, given strata P and Q of Y such that $h(P) \subseteq S$ and $g(Q) \subseteq S$, the restrictions $h_{|P} : P \to S$ and $g_{|Q} : Q \to S$ are transverse.

Lemma 4.6. *Let S be a C^1 manifold, let $f \in C^1(S,S)$, and let $h : P \to S$ and $g : Q \to S$ be C^1 maps defined on C^1 manifolds P and Q.*

If the graph Γf of f is transverse to $(h \times g)$ then $f \circ h$ is transverse to g.

Proof. See **[17]**. $\square$

Corollary 4.7. *With the hypotheses of Lemma 4.6, if $f \in Diff(S,S)$ and $h : P \hookrightarrow X$ is inclusion, then $f(P)$ is transverse to $g : Q \to S$.*

THEOREM 4.8. *Let X be a C^∞ abstract stratified set or a C^∞ (w)-regular stratified subspace of a C^∞ manifold, and let $g : Y \to X$ be a stratified map defined on a stratified space Y. For each stratified map $h : Z \to X$ there exists a deformation by isotopy h' of h in X which is transverse to g.*

Proof. By induction on the dimension of the k-skeleton A_k of A we construct a chain of n deformations by isotopy of h in $X = (A, \Sigma)$: $h = h^0 \overset{\Phi^1}{\equiv} h^1 \overset{\Phi^2}{\equiv} \cdots \overset{\Phi^n}{\equiv} h^n$ such that h^k is transverse to g in A_k for every $k = 0, \dots, n$.

Let $k > 0$ and suppose that h^{k-1} has been constructed.

By the inductive hypothesis h^{k-1} is transverse to g in A_{k-1}, so we define the restriction of h^k to A_{k-1} to coincide with h^{k-1}. We complete the proof by deforming h^{k-1} without changing it on A_{k-1}.

Step 1 : Deformation of $h^{k-1}_{|[A_k - A_{k-1}]}$. Let $\{W_\alpha\}_\alpha$ be the strata of $(h^{k-1})^{-1}(S)$, and let $h_\alpha : W_\alpha \to S$ be the restriction of h to W_α. Similarly, write $\{Y_\beta\}_\beta$ for the strata of $g^{-1}(S)$ and $g_\beta : Y_\beta \to S$ for the restriction of g to Y_β. For all α, β the set

$$T_{W_\alpha \times Y_\beta} = \big\{ f \in C^\infty(S,S) \quad \big| \quad \Gamma f \text{ is transverse to } h_\alpha \times g_\beta \big\},$$

where Γf is the graph of f, is a countable intersection of open dense sets in $C^\infty(S,S)$ **[14]** and the family $\{W_\alpha \times Y_\beta\}_{\alpha,\beta}$ being countable, so is

$$T = \bigcap_{h(W_\alpha)\subseteq S\,,\, g(Y_\beta)\subseteq S} T_{W_\alpha \times Y_\beta} = \Big\{ f \in C^\infty(S,S;C) \,\Big|\, \Gamma f \text{ transverse to } h_S \times g_S \Big\}.$$

Hence T is dense in the Baire space $C^\infty(S,S)$.

Consider now an open neighbourhood B of 1_S in $Diff(S,S)$ as in the proof of Theorem 3.1, and let $f \in B \cap T$.

Since $f \in T$, for each h_α and g_β, Lemma 4.6 implies that $f \circ h^{k-1}_\alpha$ is transverse to g_β, and so $f \circ h^{k-1}_S$ is transverse to g_S. Defining $h^k = f \circ h^{k-1}$, h^k_S is transverse to g_S.

On the other hand $f \in B$, so that the maps h^k et h^{k-1} coincide on A_{k-1}, and, by inductive hypothesis on h^{k-1}, $f \circ h^k_{A_{k-1}}$ is transverse to $g_{A_{k-1}}$ in A_{k-1}.

Therefore $f \circ h^k_{A_k}$ being transverse to g_{A_k} on $A_k = A_{k-1} \cup S$ we conclude the step 1.

Step 2 : Deformation of $h^{k-1}_{[A - A_k]}$ *and conclusion of the proof.* Consider the stratified homeomorphism $\tilde{f} = \Phi_1 = \Phi^k_1 : A \to A$ constructed in Theorem 3.1 and set

$$h^k = \tilde{f} \circ h^{k-1}.$$

Since the stratified isotopy Φ^k_t (see Theorem 3.1), satisfies $\Phi^k_1 = \tilde{f}$, $\Phi^k_{1|S} = f$ and $\Phi^k_{1|A_{k-1}} = id_{A_{k-1}}$ for the "whole deformation" h^k of h^{k-1}, $h^k = \Phi^k_1 \circ h^{k-1}$ we have :

$$h^k = \begin{cases} h^{k-1}_{A_{k-1}} & \text{on } A_{k-1}; \\ f \circ h^{k-1}_S & \text{on } S = A_k - A_{k-1}; \\ \Phi^k_1 \circ h^{k-1}_{[A - A_k]} & \text{on } A - A_k\,. \end{cases}$$

Thus we complete the proof of the theorem by setting $h' = h^n$. □

A different proof of theorem 4.8 will be given in **[10]**.

In Theorem 4.8 the deformation by isotopy $h' : Z \to X$ of h transverse to g can be obtained arbitrarily close to h. In fact since T is dense and B is a neighbourhood of 1_S in $C^\infty(S,S)$, the diffeomorphism $f \in B \cap T \subsetneq Diff(S,S)$ may be chosen arbitrarily near to 1_S and so the stratified homeomorphism $\tilde{f} = \Phi_1 : A \to A$ extension of $f : S \to S$ (see theorem 3.1) may be obtained arbitrarily close to 1_A.

Transversality of substratified objects. In analogy with the definition of Whitney substratified object V of X given by Goresky in **[8]** for V and X which are Whitney stratified ((b)-regular), we give the following definition, where (E) means a regularity conditions such as "to be an abstract stratified set", the condition (b) of Whitney, the condition (c) of Bekka, (w) of Verdier, or the Lipschitz condition of Mostowski.

Definition 4.9. Let $X = (A, \Sigma)$ be a stratified space. An *(E)-regular substratified object* of X, is a stratified space $V = (V, \Sigma_V)$, such that V is a closed subset of A, Σ_V is (E)-regular and each stratum of V is contained in a unique stratum of X. If X and V are both abstract stratified sets then we say that V is an *abstract substratified set* of X.

Obviously, if V is a substratified object of X, then the inclusion $i : V \hookrightarrow X$ is a stratified map. Moreover for every stratum S of X , $V \cap S$ is a closed subset of S with an induced stratification $V_S = \{R \text{ stratum of } V \mid R \subseteq S\}$, i.e. the restriction of V to the stratum S of X.

In what follows, when this is not ambiguous, we write simply V for a substratified object V of X. Thus V_k and A_k denote the k-skeletons V_k and X_k of V and X. Similarly, when V (or X) is an abstract stratified set equipped with a system of control data (§2), we assume given the system of control data.

Definition 4.10. Let $W = (W, \Sigma_W)$ and $W' = (W', \Sigma_{W'})$ be two substratified objects of a stratified space $X = (A, \Sigma)$. We say that *W' is a deformation by isotopy of W in A* if there exists a stratified isotopy $\Phi : A \times I \to A$ such that $\Phi_0 = 1_A$ and $W' = \Phi_1(W)$. We write then $W \overset{\Phi_t}{\equiv} W'$ or $\Phi_t : W \equiv W'$.

Of course "Deformation by isotopy" is an equivalence relation on the set of all substratified objects of X. Moreover one easily has :

Proposition 4.11. *If $\Phi : A \times I \to A$ is a stratified isotopy of $X = (A, \Sigma)$ and W is a substratified object of X , then for each $t \in I$ the image $W' = \Phi_t(W)$ is a substratified object with stratification induced by Φ_t and W' is a deformation by isotopy of W.*

Proof. We only have to remark that when W is an abstract stratified set, if

$$F_W = \{(T_{W_\alpha}, \pi_{W_\alpha}, \rho_{W_\alpha})\}_{W_\alpha \in \Sigma_W} \qquad , \qquad \begin{cases} \pi_{W_\alpha} : T_{W_\alpha} \to W_\alpha \\ \rho_{W_\alpha} : T_{W_\alpha} \to \mathbb{R} \end{cases}$$

is the system of control data for $W = \cup_\alpha W_\alpha$, then the family

$$\big\{\big(\Phi_t(T_{W_\alpha}), \Phi_t\, \pi_{W_\alpha} \Phi_t^{-1}\ ,\ \rho_{W_\alpha} \Phi_t^{-1}\big)\big\}_{\Phi_t(W_\alpha) \text{ stratum of } W'}$$

is a system of control data for $W' = \cup_\alpha \Phi_t(W_\alpha)$. In fact, all topological properties of the abstract stratified sets are preserved via the homeomorphism $\Phi_t : A \to A$ and the submersivity of the maps $(\pi_{W_\alpha}, \rho_{W_\alpha})$ is preserved by the diffeomorphisms $\{\Phi_t : X \to X\}_t$ (X stratum of A). $\square$

Suppose now that W is a substratified object of X, and that the map $h = i : W \hookrightarrow X$ is the stratified inclusion of W in X, and consider the map $h' = \Phi_1 \circ h$.

Because the transversalising deformation Φ_1 is a stratified homeomorphism, and hence is a diffeomorphism on each stratum, by Lemma 4.7 the condition *"$h' = \Phi_1 \circ h$ is transverse to g"* may be reread as *"$W' = \Phi_1(W)$ is transverse to g"*. Thus we have the following corollary which generalizes the *Transversality Lemma* of Goresky [**8**].

Corollary 4.12. *Let X be a C^∞ abstract stratified set, or a (w)-regular stratified subspace of a C^∞ manifold, and let $g : Y \to X$ be a stratified map on a stratified space Y. For each substratified object W of X and each open neighbourhood U of W in X, there exists a deformation by isotopy W' of W which is transverse to g and such that $W' \subseteq U$.*

Corollary 4.12 holds for stratifications and stratified maps more general than that of the *Transversality Lemma* of [**8**]. For we do not require :

i) that g be controlled with respect to two fixed systems of control data T_1 et T_2 respectively of Y and X or that g be the restriction of a smooth map $\tilde{g} : M_1 \to M_2$ between two smooth manifolds containing respectively Y and X;

or

ii) that W satisfy the π-fibre condition.

The π-fibre condition is a very strong restriction on the geometry of the substratified object W on X and ensures that (b)-regularity be preserved as was shown in [**8**]. Possibly other regularity conditions are preserved, for example this is the case for (a)-regularity. *A priori* there is no reason that a stratified isotopy of X as in corollary 4.12 preserve regularity conditions on substratified objects W of X other than *"being an abstract substratified set of X"*.

Corollary 4.12 (when $g : Y \hookrightarrow X$ is the inclusion map and $\dim(Y \cap S) + \dim(W \cap S) < \dim S$ for every stratum S of X) was used by M. Grinberg to prove the existence of self-indexing stratified Morse functions on complex algebraic varieties [**9**].

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