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The Steenrod p-powers in Whitney cohomology ☆

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Abstract

In his Ph.D. thesis and in the paper *Whitney stratified chains and cochains*, M. Goresky introduced, associated to each Whitney stratification X, a geometric cohomology theory $WH^*(X)$, showing that there is a bijection $R^k : WH^k(X) \to H^k(X)$. Subsequently, in 1994 I improved Goresky's theory, by first introducing a group operation in $WH^k(X)$, geometrically defined via transverse union of cochains in such a way that the representation map R^k becomes a group isomorphism, and secondly by giving a geometric construction of the Steenrod squares.

In this paper, extending techniques and results of Murolo (1994), we complete the theory by constructing geometrically (through the transversal sum), the Steenrod p-powers in the context of Whitney cohomology WH^*. A preliminary analysis explaining the Whitney cohomology of the lens spaces $L_p = S^h/\mathbb{Z}_p$ is necessary.

Keywords: Whitney stratifications; Transversality; Cohomology operations

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1. Introduction

Cohomology operations were discovered by Steenrod while solving some problems about extension of mappings between two polyhedra. Axiomatized by Cartan (squares) and Thom (p-powers) they were generalized to the category of CW-complexes again by Steenrod who introduced the homology groups of the symmetric group and gave furthermore a "small" set of generators. Dold and Nakamura showed (separately) that the cohomology operations correspond to the cohomology classes of an Eilenberg–MacLane space, so establishing the connexion between the two different approaches (then known)

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to the theory. Many important applications of cohomology operations were given and in different fields of algebraic topology, particularly by Thom to characteristic classes, sphere bundles, cobordism theory and nonembedding theorems, and with the discovery of the Adem relations in the Steenrod algebra, by Adem to fibrations on spheres and certain cohomology and homotopy groups which are not zero. The main applications were given to the stable homotopy of spheres, by Toda, who introduced some exact sequences obtaining results on the p-primary components of the groups, and by Adams, again on this subject, through his famous spectral sequence and Hopf maps. Most of these results were collected and elaborated in [14] which became the first fundamental reference.

Many papers then appeared, and about various problems: for reducing generators (Cartan and Moore), finding bases (Chow), constructing dual homology operations (Wu Ding-jia), introducing local systems of coefficients, and the question of giving concrete constructions was considered in some particular cases. Thus particular constructions of operations were given in different contexts: for simplicial spaces [16], for geometric cohomology of a ball complex [2] (through mock-bundles), for intersection homology groups of stratified topological pseudomanifolds [8], for subanalytic stratifications [9], and more recently for de Rham cohomology [3] and [4], and for cohomology of simplicial presheaves [10].

In this paper we consider the Steenrod operations in the context of the Whitney cohomology of a Whitney stratification. Recall then briefly some notions about Whitney cohomology. If X is a topological space equipped with a Whitney stratification, in [12] I improved the theory defined by Goresky [7] (which had a precursor in work of Whitney in 1947 [15] as Dold pointed out to me during the *I.C.M.* Zürich 1994) by introducing in the cohomology set $WH^k(X)$ (that I call *Whitney cohomology*) a geometric sum operation in such a way that the Goresky "representation" (bijection) $R: WH^k(X) \to H^k(X)$ becomes a group isomorphism. The set $WH^k(X)$ is a quotient set, the cocycles of which are Whitney substratified cochains of X, and in which the sum operation was defined (by me) at the representative level, via transverse union of cocycles put in general position. Furthermore, with the same transverse union operation, the pull-back map $f^*: WH^k(X_2) \to WH^k(X_1)$ induced by a stratified controlled map $f: X_1 \to X_2$ (defined acting as transverse preimage $V \to f^{-1}(V)$) turned out to be a group homomorphism. In the last section I constructed geometrically the Steenrod squares $\mathrm{Sq}^\alpha: WH^k \to WH^{k+\alpha}$ for (such) Whitney cohomology.

In this paper we complete the theory of [12] showing that a geometrical construction can be used to define the Steenrod p-powers $P^\alpha: WH^k(X) \to WH^{k+2\alpha(p-1)}(X)$ ($p > 2$ and prime) in Whitney cohomology.

Our techniques, based essentially on transversality, are well adapted to this geometric construction so that it turns out to be very simple to verify the axioms (except for $P^0 = 1$). The theory of the p-powers is in general more complicated than that of the squares, so some preliminary analysis is necessary, which constitutes the content of Section 2 and Section 3. More explicitly, in Section 2 we study Whitney cohomology $WH^*(L_p)$ of the lens space $L_p = S^h/\mathbb{Z}_p$ giving a Whitney cocycle generator of $WH^j(L_p)$ for each dimension j, while in Section 3 we examine some interesting relations about two different $\mathbb{Z}_p$-actions on the p-sphere S^p. Finally in Section 4 we define the homomorphisms $P^\alpha : WH^k(X) \to WH^{k+2\alpha(p-1)}(X)$ and show that they verify the axioms.

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Some notations

(1) All rings of coefficients (always omitted) will be the fixed field $\mathbb{Z}_p$ of the integers modulo p (p an odd prime);

(2) for each space X we associate to the product space $X^p = X \times \cdots \times X$ (p times) the free $\mathbb{Z}_p$-action A which cyclically permutes the coordinates:

$$A : X^p \to X^p, \quad A(x_1, \ldots, x_p) = (x_p, x_1, \ldots, x_{p-1});$$

(3) we consider only spheres S^h having odd dimension h, consequently there is associated to the standard embedding $S^h \hookrightarrow \mathbb{C}^{m+1}$, $m = (h-1)/2$, a natural free $\mathbb{Z}_p$-action on S^h defined by

$$T : S^h \to S^h, \quad T(z_0, \ldots, z_m) = (\lambda z_0, \ldots, \lambda z_m),$$

where $\lambda = e^{2\pi i/p}$, we then have the lens quotient space $L_p = S^h/\mathbb{Z}_p$ whose Whitney cohomology is explained in the following;

(4) the product space $X^p \times S^h$ is automatically endowed with the free $\mathbb{Z}_p$-action $A \times T$ in such a way that for every k-cocycle $V \subseteq X$ the spaces $(X^p \times S^h)/\mathbb{Z}_p \supseteq (V^p \times S^h)/\mathbb{Z}_p$ define respectively a Whitney space and one of its kp-cocycles [7,12].

If X is a Whitney object, denoting by Δ the diagonal map

$$\Delta_h : X \times L_p \to \left(X^p \times S^h\right)/\mathbb{Z}_p, \quad \Delta(x, [t]) = [x, \ldots, x, t]$$

we have the induced homomorphism $\Delta^* : WH^{kp}((X^p \times S^h)/\mathbb{Z}_p) \to WH^{kp}(X \times L_p)$.

We mean to define the p-powers P^α through the following steps:

(a) there is a well-defined map

$$\varphi : WH^k(X) \to WH^{kp}\left((X^p \times S^h)/\mathbb{Z}_p\right), \quad \varphi([V]) = \left[(V^p \times S^h)/\mathbb{Z}_p\right];$$

(b) the class

$$\Delta^*\left((V^p \times S^h)/\mathbb{Z}_p\right) \in WH^{kp}(X \times L_p) \cong \sum_{i+j=kp} WH^i(X) \otimes WH^j(L_p)$$

split by Künneth homomorphism has a component of degree $\alpha' = k + 2\alpha(p-1)$ in $WH^{\alpha'}(X)$ which we will choose in order to define $P^\alpha([V])$.

2. The Whitney cohomology of L_p

The $\mathbb{Z}_p$-action $T : S^h \to S^h$ defines a natural structure of cells on S^h (see [14, Chapter V, Section 5]) having p r-cells $\{e_i^r\}_{i \leqslant p}$ for every fixed dimension $r \leqslant h$ and each of them is transformed into any of the remaining through the (p possible) applications $T^0 = \mathrm{id}, T^1, \dots, T^{p-1}$.

Therefore the quotient space $L_p^h = S^h/\mathbb{Z}_p$ has a cellularization with a unique cell e^r in each dimension $r \leqslant h$, and it is known [14, Theorem 5.2], that if $E_0, \dots, E_h$ are the cohomology classes of the dual cochains of the cells e^r then $H^r(L_p; \mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p$ and generated by E_r:

$$H^r(L_p; \mathbb{Z}_p) \cong \langle E_r \rangle \cong \mathbb{Z}_p.$$

Moreover for the generators E_1 and E_2 the following relations hold

$$E_r = \begin{cases} E_{2n} = E_2^n & \text{if } r = 2n \text{ is even;} \\ E_{2n+1} = E_2^n \cdot E_1 & \text{if } r = 2n+1 \text{ is odd.} \end{cases}$$

Thus we can find a generator of $H^r(L_p)$ $\forall r \geqslant 1$ through their cup product.

Here we will find geometric cocycles which generate $WH^r(S^h/\mathbb{Z}_p)$ showing that they are represented by some "coordinate sphere" which respects the $\mathbb{Z}_p$-action.

Proposition 1. *The 2-cocycle $(0^2 \times S^{h-2})/\mathbb{Z}_p$ represents the class of a generator E_2 of $WH^2(S^h/\mathbb{Z}_p)$ and any other $(h-2)$-coordinate sphere of $\mathbb{C}^{m+1} = \mathbb{R}^{h+1}$ represents the same cohomology class E_2.*

Proof. We start by noting that $(0^2 \times S^1)/\mathbb{Z}_p$ is not the zero 2-cocycle in $S^3/\mathbb{Z}_p$ and that the inclusion map $I : S^h/\mathbb{Z}_p \to S^{h'}/\mathbb{Z}_p$, $I([z]) = [(z, 0)]$ induces an isomorphism in cohomology. This is easy to verify in cellular cohomology and thus through the bijection $R : WH^k \cong H^k$ (see [7] and [12]) it holds in Whitney cohomology.

Therefore using the isomorphism $I^* : WH^2(S^h/\mathbb{Z}_p) \to WH^2(S^3/\mathbb{Z}_p)$ we find by transverse preimage

$$I^*\big([(0^2 \times S^{h-2})/\mathbb{Z}_p]\big) = \big[I^{-1}\big((0^2 \times S^{h-2})/\mathbb{Z}_p\big)\big] = \big[(0^2 \times S^1)/\mathbb{Z}_p\big]$$

and then $(0^2 \times S^{h-2})/\mathbb{Z}_p$ is a cocycle generator for $WH^2(S^h/\mathbb{Z}_p)$.

Now if K is the diffeomorphism

$$K : S^h/\mathbb{Z}_p \to S^h/\mathbb{Z}_p, \quad K[z_0, \dots, z_m] = [z_m, z_0, \dots, z_{m-1}],$$

we must show (that) also $(0^2 \times S^{h-2})/\mathbb{Z}_p \equiv K^j((0^2 \times S^{h-2})/\mathbb{Z}_p)$ are cobordant cocycles, for each j, as follows by the next lemma.

Lemma. *The induced isomorphism $K^* : WH^2(S^h/\mathbb{Z}_p) \to WH^2(S^h/\mathbb{Z}_p)$ is the identity for every $m \leqslant h$. Consequently each power $K^{j*} = K^{*j}$ is the identity map.*

Proof. We prove that the cocycles

$$\begin{aligned} K^*\big((0^2 \times S^{h-2})/\mathbb{Z}_p\big) &= K^{-1}\big((0^2 \times S^{h-2})/\mathbb{Z}_p\big) \\ &= \big(S^{h-2} \times 0^2\big)/\mathbb{Z}_p \text{ and } \big(0^2 \times S^{h-2}\big)/\mathbb{Z}_p \end{aligned}$$

represent the same cohomology class showing how we can construct a $\mathbb{Z}_p$-equivariant cobordism $V : 0^2 \times S^{h-2} \equiv S^{h-2} \times 0^2$.

Let us interpret the 3-sphere S^3 as the join on $\mathbb{R}^4$

$$\begin{aligned} S^3 &\cong S^1 \times 0^2 * 0^2 \times S^1 \\ &= \{t(z_0, 0^2) + (1-t)(0^2, z_1) = (tz_0, (1-t)z_1) \mid z_0, z_1 \in S^1,\ t \in [0,1]\} \end{aligned}$$

then the homotopy

$$F: S^1 \times I \to S^3, \quad F(z,t) = (tz, (1-t)z),$$

verifying obviously $F(T\times 1) = TF$, respects the $\mathbb{Z}_p$-action, and its image $V = F(S^1 \times I)$ is just the T-equivariant cobordism between $S^1 \times 0^2$ and $0^2 \times S^1$ in S^3.

Considering similarly $S^h = S^{2m+1} \equiv S^1 * \cdots * S^1$ ($m+1$ copies), this procedure shows that $(0^2 \times S^{h-2})/\mathbb{Z}_p \equiv (S^{h-2} \times 0^2)/\mathbb{Z}_p$ are cobordant in $S^h/\mathbb{Z}_p$.

Proposition 2. *A generator $E_{2i} = E_2^i \in WH^{2i}(S^h/\mathbb{Z}_p)$ is represented by the $2i$-cocycle $(0^{2i} \times S^{h-2i})/\mathbb{Z}_p$. Moreover any other $h-2i$ coordinate sphere represents the same cohomology class. In particular $E_{h-1} = [(0^{2r} \times S^1)/\mathbb{Z}_p]$.*

Proof. With the map $K: S^h/\mathbb{Z}_p \to S^h/\mathbb{Z}_p$ (as in Proposition 1) we have

$$(0^{2i} \times S^{h-2i})/\mathbb{Z}_p = \bigcap_{j=0}^{i-1} K^j((0^2 \times S^{h-2})/\mathbb{Z}_p),$$

where intersections are two by two transversal in $S^h/\mathbb{Z}_p$.

So, since in Whitney cohomology the transversal intersection is the cup product, we find the following equalities of cohomology classes:

$$[(0^{2i} \times S^{h-2i})/\mathbb{Z}_p] = \left[\bigcap_{j=0}^{i-1} K^j((0^2 \times S^{h-2})/\mathbb{Z}_p)\right] = \prod_{j=0}^{i-1} [K^j((0^2 \times S^{h-2})/\mathbb{Z}_p)].$$

On the other hand the diffeomorphisms K^j are transversal to each cocycle and thus the last member (using also the lemma) is equal to

$$\begin{aligned} \prod_{j=0}^{i-1} [(K^{m+1-j})^{-1}((0^2 \times S^{h-2})/\mathbb{Z}_p)] &= \prod_{j=0}^{i-1} K^{m+1-j*}[(0^2 \times S^{h-2})/\mathbb{Z}_p] \\ &= \prod_{j=0}^{i-1} E_2 = E_2^i = E_{2i}. \end{aligned}$$

The second statement holds immediately because, again by the lemma, we find

$$[K^j((0^{2i} \times S^{h-2i})/\mathbb{Z}_p)] = K^{m+1-j*}([(0^{2i} \times S^{h-2i})/\mathbb{Z}_p]) = \mathrm{id}(E_{2i}) = E_{2i}.$$

3. Equivalence of $\mathbb{Z}_p$-actions on lens space

Here we show some relations between different $\mathbb{Z}_p$-actions which acting on the p-sphere S^p define the same lens subspace.

We consider the rotation map of angle $\theta = 2\pi/p$

$$T_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_1(z) = \lambda z, \quad \lambda = e^{2\pi i\theta},$$

which is an isomorphism (of order p) of the complex vector space, and so also are the product maps

$$T' = T_1 \times \cdots \times T_1 \quad (s \text{ times}) \quad \text{and} \quad Q = T_1 \times T_1^2 \times \cdots \times T_1^s$$

from $\mathbb{R}^{p-1}$ onto itself, $s = (p-1)/2$.

Denote by

$$A : \mathbb{R}^p \to \mathbb{R}^p, \quad A(x_1, \ldots, x_p) = (x_p, x_1, \ldots, x_{p-1}),$$

the other isomorphism which rotates the real coordinates; we will show that there is a diffeomorphism of quotient spaces

$$S^{p-2}/T' \cong S^{p-2}/Q \cong S_1^{p-2}/A$$

where S_1^{p-2} is a convenient $(p-2)$-sphere contained in the antidiagonal hyperplane Δ^- of $\mathbb{R}^p$.

Lemma 1. *There is a norm-preserving diffeomorphism* $G : \mathbb{R}^{p-1} \to \mathbb{R}^{p-1}$ *for which* $QG = GT'$.

Proof. Representing complex numbers with exponential notation we define G as follows: for each

$$z = (z_1, \ldots, z_s) = (\rho_1 e^{2\pi i\theta_1}, \ldots, \rho_s e^{2\pi i\theta_s}), \quad \rho_i \geqslant 0,\ \theta_i \in [0, 1[\text{ for each } i,$$

$$G(z) = G(\rho_1 e^{2\pi i\theta_1}, \ldots, \rho_s e^{2\pi i\theta_s}) = (\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i(\theta_1+\theta_2)}, \ldots, \rho_s e^{2\pi i(\theta_1+\theta_2+\cdots+\theta_s)}).$$

Then it is immediate to show that $GT' = QG$ and that G is bijective having as inverse the map

$$F(\rho_1 e^{2\pi i\theta_1}, \ldots, \rho_s e^{2\pi i\theta_s}) = (\rho_1 e^{2\pi i\theta_1}, \rho_2 e^{2\pi i(\theta_2-\theta_1)}, \ldots, \rho_s e^{2\pi i(\theta_s-\theta_{s-1})}).$$

The lemma below could be expressed in the language of group representations as the (standard fact of the) irreducible decomposition of the regular representation of $\mathbb{Z}_p$ (see [5, Section 9, Example 2]). We intend that the proof of the lemma we give is helpful in explaining the geometric meaning of those $\mathbb{Z}_p$-equivariant spaces occurring in Section 4.5 and in particular of those which we construct in the proof of Proposition 3.

Let $\Delta^- = \perp(\Delta(\mathbb{R}^p), \mathbb{R}^p)$ be the hyperplane of $\mathbb{R}^p$ orthogonal to the diagonal line $\Delta(\mathbb{R}^p)$; this is therefore a subspace equivariant with respect to the $\mathbb{Z}_p$-action A and verifies:

Lemma 2. *There is an isomorphism* $H : \mathbb{R}^{p-1} \to \Delta^-$ *of real vector spaces such that* $AH = HQ$.

Proof. Subtracting from the standard basis $e_1, \ldots, e_p$ of $\mathbb{R}^p$ its barycenter $\delta = (1/p, \ldots, 1/p)$, we have a new system of vectors $\{a_i = e_i - \delta\}$ neglecting any one of which we find a basis of Δ^- such that $A(e_i - \delta) = e_{i+1} - \delta$ or also equivalently $e_{i-1} - \delta = A^{i-1}(e_1 - \delta)$. Therefore these properties still hold for its normalized system

$$\Omega = \{u, A(u), \ldots, A^{p-1}(u)\}.$$

Let now $e = (0, 1)$ be the imaginary unit and E the vector $E = (e, \ldots, e) \in \mathbb{C}^s$.

The system $\Sigma = \{E, Q(E), \ldots, Q^{p-1}(E)\}$ verifies the same properties as Ω in such a way that we can define the isomorphism H on the basis elements setting

$$H : R^{p-1} \to \Delta^-, \quad HQ^i(E) = A^i(u) \quad \text{for each } i = 0, 1, \ldots, p-2$$

so that automatically also $HQ^{p-1}(E) = A^{p-1}(u)$ holds. It is only necessary to note that the vectors

$$E = (e, \ldots, e),\ Q(E) = (\lambda, \lambda^2, \ldots, \lambda^s), \ldots, Q^{p-2}(E) = (\lambda^{p-2}, \ldots, \lambda^{s(p-2)})$$

are linearly independent in the real vector space $\mathbb{R}^{p-1}$.

If N is the following matrix, denoting by $Z_1, \ldots, Z_s$ its complex vector columns

$$N = \begin{pmatrix} E \\ Q(E) \\ \vdots \\ Q^{p-1}(E) \end{pmatrix} = (Z_1, \ldots, Z_s) = (A_1, B_1, \ldots, A_s, B_s),$$

where $\forall i\ Z_i = (A_i, B_i)$,

we will prove the $\mathbb{R}$-independence of the $2s = p - 1$ real vectors A_i, B_i in $\mathbb{R}^{p-1}$.

The Vandermonde matrix $M = V(\lambda, \lambda^2, \ldots, \lambda^{p-1})$

$$M = \begin{pmatrix} e & e & \ldots & e & \ldots & e \\ \lambda^1 & \lambda^2 & \ldots & \lambda^s & \ldots & \lambda^{p-1} \\ \lambda^2 & \lambda^4 & \ldots & \lambda^{2s} & \ldots & \lambda^{2(p-1)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \lambda^{p-2} & \lambda^{2(p-2)} & \ldots & \lambda^{s(p-2)} & \ldots & \lambda^{(p-1)(p-2)} \end{pmatrix}$$

has the determinant $\det M = \prod_{i<j}(\lambda^i - \lambda^j) \neq 0$, thus its columns, for which we can set with slight abuse of notation

$$\begin{aligned} M &= (Z_1, \ldots, Z_s, Z_1^{-1}, \ldots, Z_s^{-1}) \\ &= ((A_1, B_1), \ldots, (A_s, B_s), (A_1, -B_1), \ldots, (A_s, -B_s)) \end{aligned}$$

are linearly independent as complex vectors.

Now every linear combination with real coefficients

$$a_1 A_1 + b_1 B_1 + \cdots + a_s A_s + b_s B_s = 0, \quad a_i, b_i \in \mathbb{R},$$

is exactly the first component of the following complex equation

$$a_1 Z_1 + \cdots + a_s Z_s + \mathrm{i} b_1 Z_1^{-1} + \cdots + \mathrm{i} b_s Z_s^{-1} = 0.$$

Thus the $\mathbb{R}$-independence of A_i, B_i follows by the $\mathbb{C}$-independence of Z_i, Z_i^{-1}.

Proposition. *If $S_1^{p-2} = H(S^{p-2})$ is the diffeomorphic image of the standard $(p-2)$-sphere in Δ^-, then we have the following diffeomorphisms of quotient spaces*

$$S^{p-2}/T' \cong S^{p-2}/Q \cong S_1^{p-2}/A.$$

Proof. This follows immediately because by the commutativity of diagrams

$$\begin{array}{ccccc} S^{p-2} & \xrightarrow{G} & S^{p-2} & \xrightarrow{H} & S_1^{p-2} \subseteq \Delta^- \\ \downarrow{T'} & & \downarrow{Q} & & \downarrow{A} \\ S^{p-2} & \xrightarrow{G} & S^{p-2} & \xrightarrow{H} & S_1^{p-2} \subseteq \Delta^- \end{array}$$

the maps G and H can be factorized, defining diffeomorphisms between the quotient spaces.

4. The p-powers P^α in $WH^*(X)$

4.1. The definition of P^α

If V is a k-cocycle of X, by the Künneth formula we have

$$\Delta_h^*\big((V^p \times S^h)/\mathbb{Z}_p\big) \in WH^{kp}(X \times L_p) \cong \sum_{i+j=kp} WH^i(X) \otimes WH^j(L_p)$$

and since every $WH^j(L_p) = \langle E_j \rangle$ is generated by E_j we can write

$$\Delta_h^*\big((V^p \times S^h)/\mathbb{Z}_p\big) = \sum_{j=0}^{kp} D_j(V) \times E_j.$$

We now want to define

$$P^\alpha(V) = D_{(k-2\alpha)(p-1)}(V),$$

i.e., the component which appears together with $E_{(k-2\alpha)(p-1)}$ in the previous factorization, namely with cohomological degree

$$\alpha' = kp - (k-2\alpha)(p-1) = k + 2\alpha(p-1).$$

Thus the map $P^\alpha : WH^k \to WH^{\alpha'}$ will be by definition the composite function $P^\alpha = \operatorname{pr} \Delta^* \varphi$

$$P^\alpha : WH^k(X) \xrightarrow{\varphi} WH^{kp}\big((X^p \times S^h)/\mathbb{Z}_p\big) \xrightarrow{\Delta^*} WH^{kp}(X \times L_p) \xrightarrow{\operatorname{pr}'_\alpha} WH^{\alpha'}(X),$$

where $\operatorname{pr} = \operatorname{pr}_{\alpha'}$ is the Gysin homomorphism

$$\operatorname{pr}_{\alpha'} : WH^{kp}(X \times L_p) \cong \sum_{i+j=kp} WH^i(X) \otimes WH^j(L_p) \to WH^{\alpha'}(X).$$

In what follows $h \in \mathbb{N}$ can be an arbitrary odd number provided $h \geqslant (k-2\alpha)(p-1)$; in the case $h < (k-2\alpha)(p-1)$ we find instead the vanishing of the designated component since $WH^{(k-2\alpha)(p-1)}(L_p^h) = 0$.

Remark. By this definition we automatically will have

$$2\alpha > k \Rightarrow (k-2\alpha)(p-1) < 0 \Rightarrow \alpha' > kp \Rightarrow P^\alpha(V) = 0,$$

i.e., one of the axioms defining the Steenrod p-powers.

Proposition 1. *The map*

$$\varphi: WH^k(X) \to WH^{kp}\big((X^p \times S^h)/\mathbb{Z}_p\big), \quad \varphi[V] = \big[(V^p \times S^h)/\mathbb{Z}_p\big],$$

is well defined.

Proof. A cobordism $\zeta: V \equiv V'$, between two cocycles V and V' of X, is a Whitney k-cochain of $X \times [0,1]$ which verifies $\zeta \cap X_\varepsilon = V \times [0,\varepsilon[\cup V' \times]1-\varepsilon, 1]$ for some small $\varepsilon > 0$, where $X_\varepsilon = X \times ([0,\varepsilon[\cup]1-\varepsilon, 1])$ [7]. Considering then the map

$$\beta: X^p \times \mathbb{R} \to [X \times \mathbb{R}]^p, \quad \beta(x_1, \ldots, x_p, t) = (x_1, t, \ldots, x_p, t),$$

the restriction of β to $X^p \times ([0,\varepsilon[\cup]1-\varepsilon, 1])$ is transverse to

$$\big(V \times [0,\varepsilon[\big)^p \cup \big(V' \times]1-\varepsilon, 1]\big)^p;$$

therefore, in the same way as in the (inductive step of the) Transversality Lemma 5.3 [7], we can deform ζ, without moving it in X_ε, in such a way to obtain a new k-cochain θ of $X \times [0,1]$ such that β is transverse to θ^p.

Since θ coincides with ζ in X_ε, then $\theta: V \equiv V'$ is a Whitney cobordism and then so is $\beta^{-1}(\theta^p): V^p \equiv V'^p$.

On the other hand, for all strata $V_1, \ldots, V_p$ of θ,

$$A^i\big(\beta^{-1}(V_1 \times \cdots \times V_p)\big) = \beta^{-1}(V_{p-i+1} \times \cdots \times V_p \times V_1 \times \cdots \times V_{p-i}) \quad \forall i = 1, \ldots, p$$

holds, by which we conclude that $\beta^{-1}(\theta^p)$ is an $\mathbb{Z}_p$-equivariant cobordism between V^p and V'^p, and then so is $(\beta^{-1}(\theta^p) \times S^h)/\mathbb{Z}_p$ between $(V^p \times S^h)/\mathbb{Z}_p$ and $(V'^p \times S^h)/\mathbb{Z}_p$.

Proposition 2. *The definition of P^α does not depend on $h \geqslant (k-2\alpha)(p-1)$.*

Proof. If $h' \geqslant h \geqslant (k-2\alpha)(p-1)$, then denoting by

$$i: S^h \to S^{h'}, \qquad I: S^h/\mathbb{Z}_p \to S^{h'}/\mathbb{Z}_p$$

respectively the natural embedding of spheres and the induced map on the quotient spaces, since $(1_{X^p} \times i)_{/\mathbb{Z}_p} \Delta_h = \Delta_{h'}(1_X \times I)$ by transversality we have

$$\begin{aligned}\Delta_h^*\big((V^p \times S^h)/\mathbb{Z}_p\big) &= \Delta_h^*\big((1_{X^p} \times i)_{/\mathbb{Z}_p}^{-1}\big((V^p \times S^{h'})/\mathbb{Z}_p\big)\big) \\ &= (1_X \times I)^* \Delta_{h'}^*\big((V^p \times S^{h'})/\mathbb{Z}_p\big).\end{aligned}$$

Thus writing the images of the maps Δ_h^* and $\Delta_{h'}^*$ through the Künneth formula and applying $1_X^* \times I^*$ to the right-hand side by the uniqueness of the splitting in

$$\sum_{i+j=kp} WH^i(X) \otimes WH^j(L_p)$$

we find

$$D^h_{(k-2\alpha)(p-1)}(V) = D^{h'}_{(k-2\alpha)(p-1)}(V)$$

(note that $E_j^h = I^*(E_j^{h'})$) as required to be proved.

We will now show that the axioms hold.

4.2. Functoriality

We recall that the nice morphisms in the context of Whitney stratifications are the *controlled maps* f [11,7], for which the induced map f^* exists in Whitney cohomology [7] and is a group homomorphism [12].

Proposition. *If $f : X \to Y$ is a controlled map between the Whitney spaces X and Y, then $P^\alpha f^* = f^* P^\alpha$.*

Proof. Defining the map

$$g : (X^p \times S^h)/\mathbb{Z}_p \to (Y^p \times S^h)/\mathbb{Z}_p, \quad g[x,\dots,x,z] = [f(x),\dots,f(x),z],$$

to each cocycle V of Y which is transverse to the map f there corresponds the cocycle $V^p \times S^h/\mathbb{Z}_p$ which is transverse to the map g, and since $g\Delta_X = \Delta_Y(f \times 1)$ we deduce then the commutativity of the diagram

$$\begin{array}{ccccccc}
WH^k(Y) & \xrightarrow{\varphi} & WH^{kp}((Y^p \times S^h)/\mathbb{Z}_p) & \xrightarrow{\Delta^*} & WH^{kp}(Y \times L_p) & \xrightarrow{\text{pr}} & WH^{\alpha'}(Y) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle (f\times 1)^*} & & \downarrow{\scriptstyle f^*} \\
WH^k(X) & \xrightarrow{\varphi} & WH^{kp}((X^p \times S^h)/\mathbb{Z}_p) & \xrightarrow{\Delta^*} & WH^{kp}(X \times L_p) & \xrightarrow{\text{pr}} & WH^{\alpha'}(X)
\end{array}$$

where $\text{pr}\,\Delta^*\varphi = P^\alpha$ so $P^\alpha f^* = f^* P^\alpha$.

4.3. Homomorphism

Proposition. *Each map $P^\alpha : WH^k(X) \to WH^{\alpha'}(X)$ is a group homomorphism.*

Proof. Since $P^\alpha = \text{pr}\,\Delta^*\varphi$ where the map pr is the Gysin homomorphism it is sufficient to show that the map $\Delta^*\varphi$ is an homomorphism too. This will be true if and only if for every pair V, V' of k-cocycles transversal in X, the relation

$$\Delta^*(((V \cup_t V')^p \times S^h)/\mathbb{Z}_p) = \Delta^*((V^p \times S^h)/\mathbb{Z}_p) + \Delta^*((V'^p \times S^h)/\mathbb{Z}_p)$$

holds.

We start showing that for the transversal union we have easily

$$(V \cup_t V')^p = V^p \cup_t V'^p \cup \bigcup_{i=1}^{p-1} (U_i * U'_{p-i})$$

where $U = \overline{V - V'}$, $U' = \overline{V' - V}$ and $U_i * U'_{p-i}$ is the union of all the products in which V appears i times and V' $(p-i)$ times. Consequently we have the equality of cocycles

$$(V \cup V')^p = V^p \cup V'^p + \bigcup_{i=1}^{p-1}(U_i * U'_{p-i}).$$

These considerations still hold after multiplication by S^h and factorization modulo $\mathbb{Z}_p$-action; therefore we find

$$\big((V \cup V')^p \times S^h\big)/\mathbb{Z}_p = \big((V^p \cup_t V'^p) \times S^h\big)/\mathbb{Z}_p + \left(\bigcup_{i=1}^{p-1}(U_i * U'_{p-i}) \times S^h\right)/\mathbb{Z}_p.$$

Now considering the new stratification (refinement) of $X^p \times S^h/\mathbb{Z}_p$ given by

$$\big[X^p \times S^h/\mathbb{Z}_p\big]' := \big((X^p - d(X)) \times S^h\big)/\mathbb{Z}_p \cup \big(d(X) \times S^h\big)/\mathbb{Z}_p,$$

i.e., in which for each stratum S of X the "diagonal submanifold" $(d(X) \times S^h)/\mathbb{Z}_p$ is considered as a new stratum, where $d: X \to X^p$ is the diagonal map, we have the commutative diagram

$$\begin{array}{ccc} X \times L_p & \xrightarrow{\Delta'} & \big[(X^p \times S^h)/\mathbb{Z}_p\big]' \\ & \underset{\Delta}{\searrow} & \Big\downarrow I = \mathrm{id} \\ & & \big(X^p \times S^h\big)/\mathbb{Z}_p \end{array}$$

and so

$$\begin{aligned} \Delta^*\left(\left(\bigcup_{i=1}^{p-1}(U_i * U'_{p-i}) \times S^h\right)/\mathbb{Z}_p\right) &= \Delta'^* I^*\left(\left(\bigcup_{i=1}^{p-1}(U_i * U'_{p-i}) \times S^h\right)/\mathbb{Z}_p\right) \\ &= \Delta'^*\left(\left(\left(\bigcup_{i=1}^{p-1}(U_i * U'_{p-i})\right) \times S^h\right)/\mathbb{Z}_p\right) \\ &= \left[d^{-1}\left(\bigcup_{i=1}^{p-1}(U_i * U'_{p-i})\right)\right] \times L_p = 0, \end{aligned}$$

where for each i, $d^{-1}(U_i * U'_{p-i})$ = union of $\binom{p}{i}$ cochains which are cobordant in $X = 0$ $(\mathrm{mod}\, p)$.

Thus we can conclude since

$$\begin{aligned} \Delta^*\big(\big((V \cup_t V')^p \times S^h\big)/\mathbb{Z}_p\big) &= \Delta^*\big(\big((V^p \cup_t V'^p) \times S^h\big)/\mathbb{Z}_p\big) \\ &\quad + \Delta^*\left(\left(\bigcup_i (U_i * U'_{p-i}) \times S^h\right)/\mathbb{Z}_p\right) \end{aligned}$$

$$\begin{aligned}&= \Delta^*\big(\big(\big(V^p \cup_t V'^p\big) \times S^h\big)/\mathbb{Z}_p\big)\\ &= \Delta^*\big(\big(V^p \times S^h\big)/\mathbb{Z}_p \cup_t \big(V'^p \times S^h\big)/\mathbb{Z}_p\big)\\ &= \Delta^*\big(\big(V^p \times S^h\big)/\mathbb{Z}_p\big) + \Delta^*\big(\big(V'^p \times S^h\big)/\mathbb{Z}_p\big).\end{aligned}$$

4.4. Cup product

Proposition. *If V is a k-cocycle of X and $k = 2\alpha$, then*

$$P^\alpha(V) = [V] \cup [V] \cup \cdots \cup [V]$$

is the cup-product p times of the class $[V]$.

Proof. With the diagonal maps d and Δ, and the projections q and p of the commutative diagram

$$\begin{array}{ccc} X \times S^h & \xrightarrow{d\times 1} & X^p \times S^h \\ \Big\downarrow{\scriptstyle 1\times q} & & \Big\downarrow{\scriptstyle p} \\ X \times (S^h/\mathbb{Z}_p) & \xrightarrow{\Delta} & (X^p \times S^h)/\mathbb{Z}_p \end{array}$$

we have $(1 \times q)^*\Delta^* = (d \times 1)^*p^*$ in Whitney cohomology.

Since the projection p is a local diffeomorphism (when restricted to each stratum) and so is transversal to the cocycle $(V^p \times S^h)/\mathbb{Z}_p$, we immediately have

$$p^*\big(\big[\big(V^p \times S^h\big)/\mathbb{Z}_p\big]\big) = \big[p^{-1}\big(\big(V^p \times S^h\big)/\mathbb{Z}_p\big)\big] = \big[V^p \times S^h\big].$$

Then

$$\begin{aligned}(1 \times q)^*\Delta^*\big(\big(V^p \times S^h\big)/\mathbb{Z}_p\big) &= (d \times 1)^*p^*\big(\big(V^p \times S^h\big)/\mathbb{Z}_p\big)\\ &= (d \times 1)^*\big[\big(V^p \times S^h\big)/\mathbb{Z}_p\big]\\ &= d^* \times 1^*_{S^h}\big([V] \times \cdots \times [V] \times [S^h]\big)\\ &= d^*\big([V] \times \cdots \times [V]\big) \times [S^h],\end{aligned}$$

in such a way that expanding in the left-hand side the image of Δ^* by Künneth's formula, and applying to it $(1 \times q)^* = 1^* \times q^*$ we find that the only nonzero term must necessarily be the one having the component of degree $kp = 2\alpha p$ in X.

On the other hand this component is by definition just $D_0([V])$, thus we conclude that

$$\begin{aligned}P^\alpha\big([V]\big) &= D_{(k-2\alpha)(p-1)}\big([V]\big) = D_0\big([V]\big)\\ &= d^*\big([V] \times \cdots \times [V]\big) = [V] \cup \cdots \cup [V] \ (p \text{ times}).\end{aligned}$$

4.5. P^0 = identity

Proposition 1. *If X is a Whitney space, then $P^0 = 1_{WH^k(X)}$ is the identity map, for each k.*

Proof. This proof is given automatically substituting in 5.4 [12] the squares Sq^0 with the maps P^0, for all spaces X, Y, X', X_{n-1} under consideration.

Now when $\dim X = 0$ the equality

$$\begin{aligned}\Delta^*\big(\big[\big((P,\dots,P)\times S^h\big)/\mathbb{Z}_p\big]\big) &= \big[\Delta^{-1}\big(\big((P,\dots,P)\times S^h\big)/\mathbb{Z}_p\big)\big]\\ &= \big[P\times\big(S^h/\mathbb{Z}_p\big)\big] = [P]\times[L_p]\end{aligned}$$

implies $D^0([P]) = [P]$ and so $P^0([P]) = [P]$ (Step 3).

Therefore also in this case we reduce ourselves only to show that $P^0([P]) = [P]$ when $\dim X = n$, and again this is shown in the following separate proposition.

Proposition 2. *If P is a point in any n-cell C^n of X, then*

$$\Delta^*\big(\big[\big((P,\dots,P)\times S^h\big)/\mathbb{Z}_p\big]\big) = [P]\times E_{n(p-1)}$$

and in particular $P^0([P]) = [P]$.

Proof. Let us identify C^n with $\mathbb{R}^n$.

Step 1. Denote with $\Delta = \{(x,\dots,x) \mid x\in\mathbb{R}^n\}$ the diagonal space of $(\mathbb{R}^n)^p$ and by $U^- = \perp(\Delta,\mathbb{R}^{np})$ its orthogonal complement (having dimension $n(p-1)$); we find first that these spaces are A-invariant with respect to the $\mathbb{Z}_p$-action

$$A:(\mathbb{R}^n)^p\to(\mathbb{R}^n)^p,\quad A(x_1,\dots,x_p) = (x_p,x_1,\dots,x_{p-1})$$

which rotates the coordinates and that we have a natural diffeomorphism G of manifolds and submanifolds

$$\big((\mathbb{R}^n)^p\times S^h\big)/\mathbb{Z}_p \cong \big((\Delta\times U^-)\times S^h\big)/\mathbb{Z}_p \cong \Delta\times\big((U^-\times S^h)/\mathbb{Z}_p\big),$$

$$\begin{aligned}\big((P,\dots,P)\times S^h\big)/Z_p &\cong \big((P,\dots,P)\times(0,\dots,0)\times S^h\big)/\mathbb{Z}_p\\ &\cong (P,\dots,P)\times\big((0^p\times S^h)/\mathbb{Z}_p\big).\end{aligned}$$

Writing $I: S^h/\mathbb{Z}_p\to(U^-\times S^h)/\mathbb{Z}_p$ for the 0-section embedding we have a commutative diagram

$$\begin{array}{ccc}\mathbb{R}^n\times L_p & \xrightarrow{\ \Delta\ } & \big((\mathbb{R}^n)^p\times S^h\big)/\mathbb{Z}_p\\ & \searrow^{d\times I} & \downarrow G\\ & & \Delta\times\big((U^-\times S^h)/\mathbb{Z}_p\big)\end{array}$$

in such a way that $\Delta^*G^* = (d\times 1)^*$ in Whitney cohomology.

Hence

$$\begin{aligned}\Delta^*\big(\big[\big((P,\dots,P)\times S^h\big)/\mathbb{Z}_p\big]\big) &= \Delta^*G^*\big(\big[(P,\dots,P)\times\big((0^p\times S^h)/\mathbb{Z}_p\big)\big]\big)\\ &= (d\times I)^*\big(\big[(P,\dots,P)\times\big((0^p\times S^h)/\mathbb{Z}_p\big)\big]\big)\\ &= d^*\big(\big[(P,\dots,P)\big]\big)\times I^*\big(\big[(0^p\times S^h)/\mathbb{Z}_p\big]\big)\\ &= [P]\times I^*\big(\big[(0^p\times S^h)/\mathbb{Z}_p\big]\big)\end{aligned}$$

the last equality holding by transversal preimage through the map $d:\mathbb{R}^n\xrightarrow{\cong}\Delta$.

Step 2. $I^*([(0^{np} \times S^h)/\mathbb{Z}_p]) = E_{n(p-1)}$, i.e., the former is just the generator of $WH^{n(p-1)}(L_p)$ described in Section 2.

If we set then:

$$U_n = U^- = \perp(\Delta(\mathbb{R}^n), \mathbb{R}^{np}) = \{(x_1, \dots, x_p) \in \mathbb{R}^{np} \text{ orthogonal to each } (x, \dots, x) \in \mathbb{R}^{np}\},$$

$$U_1 = \Delta^- = \perp(\Delta(\mathbb{R}), \mathbb{R}^p) = \{(\xi^1, \dots, \xi^p) \in \mathbb{R}^p \text{ orthogonal to each } (\xi, \dots, \xi) \in \mathbb{R}^p\}$$

we easily find a natural isomorphism

$$g: U_n \to U_1 \times \dots \times U_1, \quad g(x_1, \dots, x_p) = ((\xi_1^1, \dots, \xi_p^1), \dots, (\xi_1^n, \dots, \xi_p^n)),$$

(where $x_i = (\xi_i^1, \dots, \xi_i^n)$) which can be quotiented verifying:

(1) the commutativity of the diagram

$$\begin{array}{ccc} L_p = S^h/\mathbb{Z}_p & \xrightarrow{I = I_n} & (U_n \times S^h)/\mathbb{Z}_p \\ \downarrow d & & \downarrow g \\ S^h/\mathbb{Z}_p \times \dots \times S^h/\mathbb{Z}_p & \xrightarrow{\times_p \text{ times} I_1} & (U_1 \times S^h)/\mathbb{Z}_p \times \dots \times (U_1 \times S^h)/\mathbb{Z}_p \end{array}$$

(2) $g^*([(0^p \times S^h)/\mathbb{Z}_p \times \dots \times (0^p \times S^h)/\mathbb{Z}_p]) = [(0^{np} \times S^h)/\mathbb{Z}_p]$ directly through transversal preimage.

Thanks to this and since $I_1^*([(0^p \times S^h)/\mathbb{Z}_p]) = E_{p-1}$ (see next Proposition 3) we conclude that

$$\begin{aligned} I_n^*([(0^{np} \times S^h)/\mathbb{Z}_p]) &= I_n^* g^*([(0^p \times S^h)/\mathbb{Z}_p \times \dots \times (0^p \times S^h)/\mathbb{Z}_p]) \\ &= d^*(I_1^* \times \dots \times I_1^*)([(0^p \times S^h)/\mathbb{Z}_p \times \dots \times (0^p \times S^h)/\mathbb{Z}_p]) \\ &= d^*(I_1^*([(0^p \times S^h)/\mathbb{Z}_p]) \times \dots \times I_1^*([(0^p \times S^h)/\mathbb{Z}_p])) \\ &= d^*(E_{p-1} \times \dots \times E_{p-1}) \\ &= E_{p-1} \cup \dots \cup E_{p-1} = E_{p-1}^n = E_{n(p-1)}. \end{aligned}$$

Proposition 3. *The* 0*-section embedding* $I_1 : S^h/\mathbb{Z}_p \to (\Delta^- \times S^h)/\mathbb{Z}_p$ *verifies the relation* $I_1^*([(0^p \times S^h)/\mathbb{Z}_p]) = E_{p-1}$.

Proof. First of all we observe that we can suppose $h = p$ and $I_1 = I_1^p : S^p/\mathbb{Z}_p \to (\Delta^- \times S^p)/\mathbb{Z}_p$. In fact by the commutative diagram of embeddings

$$\begin{array}{ccc} S^p/\mathbb{Z}_p & \xrightarrow{I_1^p} & (\Delta^- \times S^p)/\mathbb{Z}_p \\ \downarrow I & & \downarrow J \\ S^h/\mathbb{Z}_p & \xrightarrow{I_1^h} & (\Delta^- \times S^h)/\mathbb{Z}_p \end{array}$$

we find $I_1^{p*} J^* = I^* I_1^{h*}$ where J^* verifies also

$$J^*([(0^p \times S^h)/\mathbb{Z}_p]) = [J^{-1}((0^p \times S^h)/\mathbb{Z}_p)] = [(0^p \times S^p)/\mathbb{Z}_p]$$

directly by transverse preimage.

Therefore

$$I^* I_1^{h*}\big(\big[(0^p \times S^h)/\mathbb{Z}_p\big]\big) = I_1^{p*} J^*\big(\big[(0^p \times S^h)/\mathbb{Z}_p\big]\big) = I_1^{p*}\big(\big[(0^p \times S^p)/\mathbb{Z}_p\big]\big)$$

and so we can conclude because I^* is an isomorphism verifying $I^*(E^h_{p-1}) = E^p_{p-1}$.

Now we will prove

$$I_1^*\big(\big[(0^p \times S^p)/\mathbb{Z}_p\big]\big) = E_{p-1}, \quad \text{i.e.,}\ I_1^*\big(\big[(0^p \times S^p)/A \times T\big]\big) = \big[(0^{2s} \times S^1)/T\big]$$

(where T is the $\mathbb{Z}_p$-action defined in the introduction with $h = p$) directly determining a cocycle U which is cobordant to $(0^p \times S^p)/\mathbb{Z}_p$, transversal to I_1 and such that $I_1^{-1}(U) = (0^{2s} \times S^1)/\mathbb{Z}_p$.

Equivalently we shall determine a p-sphere $V \subseteq \Delta^- \times S^p$ which is $A \times T$-equivariant in such a way that

(a) $V \equiv 0^p \times S^p$ with an $(A \times T)$-equivariant cobordism;

(b) $V \perp 0^p \times S^p$ (i.e., they are transversal in $\Delta^- \times S^p$) and

$$V \cap 0^p \times S^p = 0^p \times 0^{2s} \times S^1.$$

Let then $H : \mathbb{R}^{p-1} \to \Delta^-$ be the diffeomorphism with which $AH = HT'$ (see Section 3 and observe that by $h = p$ then $T = T' \times T_1$ holds), and G the diagonal immersion defined on each $z' = (z_0, \ldots, z_{s-1}) \in C^s = \mathbb{R}^{p-1}$ by

$$G : \mathbb{R}^{p-1} \to \Delta^- \times \mathbb{R}^{p-1}, \quad G(z') = \big(H(z'), z'\big).$$

It is easy to verify that the map $F = G \times 1_{\mathbb{R}^2}$ and its restriction $F : S^p \to S_1^{p-2} \times S^p$ are again $\mathbb{Z}_p$-equivariant: i.e., $FT = (A \times T)F$.

Now the "diagonal movement" performed by the map F on $S^p \cong 0^p \times S^p$ is such that the p-sphere image $V = F(S^p)$ is clearly $A \times T$-equivariant

$$(A \times T)(V) = (A \times T)\big(F(S^p)\big) = FT(S^p) = F\big(S^p\big) = V.$$

In this way considering the homotopy $B : S^p \times I \to \Delta^- \times S^p$,

$$B(z,t) = tF(z) + (1-t)(0^p, z) = \big(tH(z'), tz + (1-t)z\big) = \big(tH(z'), z\big)$$

which again respects the $\mathbb{Z}_p$-action, the image $B(S^p \times I)$ is the cobordism between V and $0^p \times S^p$ as required in (a).

It is then easy to verify the condition of transversality and intersection (b).

In other terms this means that V is transversal to $I_1 : S^p \to \Delta^- \times S^p$ and $I_1^{-1}(V) = 0^{2m} \times S^1$.

Finally we conclude that

$$I_1^*\big(\big[(0^p \times S^p)/\mathbb{Z}_p\big]\big) = I_1^*\big([V/\mathbb{Z}_p]\big) = \big[I_1^{-1}(V/\mathbb{Z}_p)\big] = \big[(0^{2s} \times S^1)/\mathbb{Z}_p\big] = E_{p-1}.$$

4.6. *The Cartan formula*

We start showing that in the Künneth expansion

$$\Delta^*\big((V^p \times S^h)/\mathbb{Z}_p\big) = \sum_{j=0}^{kp} D_j(V) \times E_j$$

the only nonzero components $D_j(V)$ are exactly those we selected to define the Steenrod powers $P^\alpha(V)$.

The characterization of $\Delta^*(((P,\dots,P)\times S^h)/\mathbb{Z}_p)$ (see 4.5) implies the following

Proposition 1. *For all $j > k(p-1)$ each map $D_j : WH^k(X) \to WH^{kp-j}(X)$ is the null homomorphism. Then*

$$\Delta^*\big((V^p \times S^h)/\mathbb{Z}_p\big) = \sum_{j=0}^{k(p-1)} D_j(V) \times E_j.$$

Proof. This is identical to Proposition 3 of 5.4 [12], where we need only substitute the maps R^r_X and $R^r_{X_k}$ with the $D_{j,X}$ and D_{j,X_k} and the numbers $r \leqslant k$ with $j \geqslant k(p-1)$.

Proposition 2. *If j is even, then D_j (defined in $WH^k(X)$) is nonzero only if $j = (k-2\alpha)(p-1)$ for some α.*

Proof. Let f be the map on S^h $(h = 2m+1)$, defined through the $m+1$ complex coordinates of $z = (z_0,\dots,z_m) = (\rho_0 e^{2\pi i\theta_0},\dots,\rho_m e^{2\pi i\theta_m}) \in S^h$ by

$$f : S^h \to S^h, \quad f(z_0,\dots,z_m) = \big(\rho_0 e^{2\pi i q\theta_0},\dots,\rho_m e^{2\pi i q\theta_m}\big)$$

and $T : S^h \to S^h$ the $\mathbb{Z}_p$-action on S^h of Section 1.

Since it is immediately verified that $fT = T^p f$, then f can be factorized defining a map on the lens space $L_p = S^h/\mathbb{Z}_p$

$$g : S^h/\mathbb{Z}_p \to S^h/\mathbb{Z}_p, \quad g[z] = \big[f(z)\big].$$

Now recalling that each group $WH^j(L_p) = \langle E_j\rangle \cong \mathbb{Z}_p$ has the generator

$$E_j = \begin{cases} E_2^i & \text{if } j = 2i \text{ is even,} \\ E_2^i \cdot E_1 & \text{if } j = 2i+1 \text{ is odd,} \end{cases}$$

we have $g^*(E_2) = qE_2$ in $WH^2(L_p)$ for some generator q of the multiplicative group $\mathbb{Z}_p^* = \mathbb{Z}_p - \{0\}$, and so we find that for all $j = 2i$ even, $g^*(E_j) = q^i E_j$ holds.

On the other hand by the commutative diagram

$$\begin{array}{ccc} X \times L_p & \xrightarrow{1\times g} & X \times L_p \\ \Delta\big\downarrow & & \big\downarrow\Delta \\ (X^p \times S^h)/\mathbb{Z}_p & \xrightarrow{[1\times f]} & (X^p \times S^h)/\mathbb{Z}_p \end{array}$$

we find $(1\times g)^*\Delta^* = \Delta^*[1\times f]^*$, with $[1\times f]$ a local diffeomorphism verifying the transverse preimage relation of sets $[1\times f]^{-1}((V^p \times S^h)/\mathbb{Z}_p) = (V^p \times S^h)/\mathbb{Z}_p$ for all k-cocycles V of X.

Then we can write

$$[1\times f]^*\big(\big[(V^p \times S^h)/\mathbb{Z}_p\big]\big) = \beta\big[(V^p \times S^h)/\mathbb{Z}_p\big]$$

where β is the multiplicity of preimage, for which we will show that $\beta = q^{ks}$.

In fact substituting in

$$(1 \times g)^* \Delta^*\big((V^p \times S^h)/\mathbb{Z}_p\big) = \beta \Delta^*\big((V^p \times S^h)/\mathbb{Z}_p\big)$$

the formula of the previous lemma

$$\Delta^*\big((V^p \times S^h)/\mathbb{Z}_p\big) = \sum_{j=0}^{k(p-1)} D_j(V) \times E_j$$

we immediately have

$$\sum_{j=0}^{k(p-1)} D_j(V) \times g^*(E_j) = \sum_{j=0}^{k(p-1)} D_j(V) \times \beta E_j$$

and thus, for each j

$$D_j(V) \times g^*(E_j) = D_j(V) \times \beta E_j.$$

Now the last relation can hold only when both members are null and hence $D_j(V) = 0$ or $D_j(V) \neq 0$ and $g^*(E_j) = \beta E_j$. Therefore since $P^0 = \mathrm{id}$ and hence $D^{(k)}_{k(p-1)}(V) \neq 0$ we deduce that

$$\beta E_{k(p-1)} = g^*\big(E_{k(p-1)}\big) = g^*(E_{2ks}) = q^{ks} E_{2ks} = q^{ks} E_{k(p-1)}$$

and so $\beta = q^{ks}$.

Finally if $j = 2i$ is even, by $g^*(E_j) = q^{ks} E_j$ we conclude that the only $D^{(k)}_j(V)$ not necessarily zero occur when

$$q^i \equiv q^{ks} (\operatorname{mod} p) \Leftrightarrow q^{ks-i} \equiv 1 (\operatorname{mod} p) \Leftrightarrow ks - i = \alpha(p-1) \Leftrightarrow i = ks - \alpha(p-1)$$

for some integer α, i.e., just when

$$j = 2i = 2ks - 2\alpha(p-1) = k(p-1) - 2\alpha(p-1) = (k - 2\alpha)(p-1).$$

Remark. We could similarly show for completeness (although this is not necessary) that for j odd $D_j(V)$ is not necessarily zero only when $j = (k-2\alpha)(p-1) - 1$, for some integer α.

Theorem. *For every $V \in WH^a(X)$, every $V' \in WH^b(X)$ and for all i we have*

$$P^i(V \times V') = \sum_{\alpha+\beta=i} P^\alpha(V) \times P^\beta(V').$$

Proof. Denoting by $f, g, \Delta, d, \mathrm{Inv}$ the following diagonal maps

$$f : X^2 \times L_p \to (X \times L_p)^2, \quad f\big(x, x', [z]\big) = \big(x, [z], x', [z]\big),$$

$$g : \big((X^2)^p \times S^h\big)/\mathbb{Z}_p \to \big((X^p \times S^h)/\mathbb{Z}_p\big)^2,$$
$$g[x_1, x'_1, \ldots, x_p, x'_p, z] = \big([x_1, \ldots, x_p, z], [x'_1, \ldots, x'_p, z]\big),$$

$$\Delta' = \Delta_{X^2} : X^2 \times L_p \to \left((X^2)^p \times S^h\right)/\mathbb{Z}_p,$$
$$\Delta'\left(x, x', [z]\right) = \left[(x, x'), \ldots, (x, x'), z\right],$$
$$d : L_p \to (L_p)^2, \quad d\left([z]\right) = \left([z], [z]\right),$$
$$\mathrm{Inv} : X^2 \times (L_p)^2 \to (X \times L_p)^2, \quad \mathrm{Inv}\left(x, x', [z], [z']\right) = \left(x, [z], x', [z']\right),$$

it is easy to see that

(a) $(\Delta \times \Delta) f = g\Delta'$ and consequently $f^*(\Delta \times \Delta)^* = \Delta'^* g^*$;

(b) $f = \mathrm{Inv}\,(1_{X^2} \times d)$ and similarly $f^* = (1 \times d)^* \mathrm{Inv}^*$;

(c) $g^*((V^p \times S^h)/\mathbb{Z}_p \times (V'^p \times S^h)/\mathbb{Z}_p) = ((V \times V')^p \times S^h)/\mathbb{Z}_p$ which holds directly through transverse preimage.

So first of all we have

$$\begin{aligned}\Delta'^*\left(\left((V \times V')^p \times S^h\right)/\mathbb{Z}_p\right) &= \Delta'^* g^*\left(\left(V^p \times S^h\right)/\mathbb{Z}_p \times \left(V'^p \times S^h\right)/\mathbb{Z}_p\right)\\ &= f^*\left(\Delta^*\left(\left(V^p \times S^h\right)/\mathbb{Z}_p\right) \times \Delta^*\left(\left(V'^p \times S^h\right)/\mathbb{Z}_p\right)\right).\end{aligned}$$

Then by the Künneth formula, expanding both images of Δ^*

$$\Delta^*\left(\left(V^p \times S^h\right)/\mathbb{Z}_p\right) = \sum_{j=0}^{a(p-1)} D_j(V) \times E_j,$$

$$\Delta^*\left(\left(V'^p \times S^h\right)/\mathbb{Z}_p\right) = \sum_{l=0}^{b(p-1)} D_l(V') \times E_l,$$

we also find

$$\begin{aligned}&f^*\left(\Delta^*\left(\left(V^p \times S^h\right)/\mathbb{Z}_p\right) \times \Delta^*\left(\left(V'^p \times S^h\right)/\mathbb{Z}_p\right)\right)\\ &\quad= \sum_{j=0}^{a(p-1)} \sum_{l=0}^{b(p-1)} (1 \times d)^* \mathrm{Inv}^*\left(D_j(V) \times E_j \times D_l(V') \times E_l\right)\\ &\quad= \sum_{\nu=0}^{(a+b)(p-1)} \sum_{j+l=\nu} (-1)^{j(bp-l)} D_j(V) \times D_l(V') \times d^*(E_j \times E_l).\end{aligned}$$

Now recalling that all cocycles E_j are generated by the cup products of the single cocycles E_2 and E_1 (with $E_1^2 = 0$) and that:

$$d^*(E_j \times E_l) = E_j \cup E_l = \begin{cases} 0 & \text{if } j \text{ and } l \text{ are both odd;}\\ E_{j+l} & \text{otherwise;}\end{cases}$$

we deduce

$$\Delta'^*\left(\left((V \times V')^p \times S^h\right)/\mathbb{Z}_p\right) = \sum_{\nu=0}^{(a+b)(p-1)} \sum_{\substack{j+l=\nu\\ j,l \text{ not both odd}}} D_j(V) \times D_l(V') \times E_{j+l}.$$

Therefore if we consider the index $\nu = (a+b-2i)(p-1)$, this is even, ensuring that so are both j and l; thus the only $D_j(V)$ and $D_l(V')$ not necessarily zero (by Proposition 1) are of the type

$$D_{(a-2\alpha)(p-1)}(V), \quad D_{(b-2\beta)(p-1)}(V'), \quad \text{with } \alpha + \beta = i.$$

Finally, expanding also $\Delta'^*(((V \times V')^p \times S^h)/\mathbb{Z}_p)$ by the Künneth formula, and observing that $(-1)^{j(bp-l)} = 1$ because now j is even we conclude that

$$D_\nu(V \times V') = \sum_{\alpha+\beta=i} D_{(a-2\alpha)(p-1)}(V) \times D_{(b-2\beta)(p-1)}(V'),$$

i.e.,

$$P^i(V \times V') = \sum_{\alpha+\beta=i} P^\alpha(V) \times P^\beta(V').$$

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