Ricerche di Matematica
Vol. XLIII , fasc. 2°, (1994), 175-204

Whitney homology, cohomology and Steenrod squares

CLAUDIO MUROLO (*)

ABSTRACT. – In [1] GORESKY proved that, for a Whitney object X, there is a bijection $R^k : WH^k(X) \to H^k(X)$ where $WH^k(X)$ is a set obtained from stratified cochains in X by passing to the quotient with respect to a cobordism relation. In this paper ideas and techniques contained in [1] are expanded by first of all introducing a group operation in $WH^k(X)$, geometrically defined via transerse union of cochains, in such a way that the representation map R^k becomes a group isomorphism, and secondly, using transverse union, by giving a geometrical construction of Steenrod squares in the context of Whitney cochains.

KEY WORDS: *Whitney stratifications, transversality, cohomology theory, Steenrod squares.*

A.M.S. CLASSIFICATION: 57R95, 57N80, 55N20, 55M05

1. Introduction

As is known, a Whitney object X in a smooth manifold M is a set stratified by submanifolds of M, in which every pair of strata verifies Whitney's condition B. In [1] GORESKY lays foundations for a geometric description of homology and cohomology in the context of Whitney chains, which he also calls geometric chains. Let us briefly say that the notion of Whitney chain is the exact analogue of simplicial chain and is obtained from the latter by substituting "simplex" by "stratum" and "simplicial complex" by "Whitney object". Two Whitney cycles are called cobordant if they are

(*) Dipartimento di Matematica e Applicazioni "R. Caccioppoli", via Claudio 21, 80125 Napoli, Italia

L-equivalent in the sense of Thom, i.e. if there exists a Whitney chain in $X \times [0,1]$ whose boundary is the difference of the two cycles placed one in $X \times 0$ and the other in $X \times 1$. Goresky thus constructs, for every $k \geq 0$ and in the usual way, a quotient set $WH_k(X)$ to which we can refer as the (k-dimensional) Whitney homology of X. Let us note here that Goresky does not introduce a sum operation in $WH_k(X)$. On the other hand, since each Whitney cycle admits a fundamental class in singular homology, he deduces the existence of a map between sets $R_X : WH_k(X) \to H_k(X)$, completely analogous to the representation map of Thom-Steenrod between the differential bordism and singular homology of a space. One of the main results of Goresky asserts that if X is a manifold then R_X is a bijective map.

The cohomological case is surprisingly simpler and more fruitful, after having established the definitions with care. A (k-dimensional) Whitney cochain in X is essentially a Whitney object embedded in X (with codimension k) in such a way that it "meets transversally" the singularities of X.

Goresky introduces a coboundary operator and shows that there is a smooth neighborhood U of X in M such that each Whitney cocycle of X is obtained by transverse intersection of X itself with a Whitney cocycle of U modulo ∂U. We recognize in this last construction the germ of the Alexander Duality for the embedding of X in M, and so it is not surprising that the bijectivity of R_U in homology (U is a manifold!) entails bijectivity in cohomology for the map $R_X : WH^k(X) \to H^k(X)$ whatever X.

In this paper we want to complete Goresky's paper, transforming R_X from a simple bijective correspondence of sets to an equivalence of cohomological theories. This is obtained by introducing in $WH^k(X)$ a group operation geometrically defined through "transverse union of cocycles" and showing that in such a way, R becomes a group isomorphism, or better still a natural transformation of functors. In fact if $f : X_1 \to X_2$ is a controlled map between the Whitney objects X_1 and X_2, then $f^* : WH^k(X_2) \to WH^k(X_1)$ defined through "transverse preimage" turns out to be a group homomorphism. The whole theory is developed also with coefficients in an arbitrary abelian group G. This is precisely the content of § 3; while in § 2 we treat the homological case of Whitney homology of a smooth manifold X.

It seems hard, at the moment, to define a "transverse union" of cycles in an arbitrary Whitney object X, since in contrast to cocycles (which meet transversally the singularities of X) it is not clear, if and in what way it may be possible to put two cycles of X in a position of mutual transversality.

This is the same obstacle that limits to a smooth manifold the representation theorem of GORESKY in the homological case.

In the second part of the paper we apply the operation of transverse union together with the deformation techniques developed by Goresky in order to construct Steenrod squares using Whitney cochains instead of singular cochains. The method is the well-known one of the quadratic construction $WH^k(X) \to WH^{2k}(X^2 \times S^h/Z_2)$, but it needs particular care to ensure that the Whitney condition B is preserved during the process.

2. Whitney homology

A Whitney object (or space) X in a manifold M is a pair (X, ζ) where X is a closed subset of M and $\zeta := \{X^i_j | j \in J_i \text{ and } i \leq n\}$ a stratification of X, i.e. a locally finite partition of smooth connected manifolds, called strata ($\dim X^j_i = i$) such that

a) the h-skeleton $X_h := \bigcup\limits_{j \in J_i} \bigcup\limits_{i \leq h} X^j_i$ is closed in X (and in M);

b) each $\{X^i_j\}_{j \in J_i}$ is the family of all connected components of $X_i - X_{i-1}$;

c) each pair of distinct strata verifies Whitney's condition B (W.C.B see [2]).

For every Whitney space X a fixed stratification will be understood. In such a way for each $h \leq \dim X = n$ the h-skeleton X_h is again a (h-dimensional) Whitney space with the obvious induced stratification. Mather proved in [2] that every Whitney space X admits a system of control data (S.C.D.) and that two strata A, B with $A \cap \overline{B} \neq \emptyset$ verify both $\dim A < \dim B$ and $A \subseteq \overline{B}$ (we then write $A < B$). Thus we immediately have $\overline{B} = \bigsqcup\limits_{A \leq B} A$ and $\overline{B} - B = \bigsqcup\limits_{A < B} A$ ($\bigsqcup =$ disjoint union).

2.1 – Remarks about the Goresky representation

A stratified k-subspace of X is a Whitney object V of dimension k which has every stratum A contained in some stratum S_A of X. A k-orientation of V is an element $z = \sum\limits_{j \in J_k} n_j V^k_j$ of the free abelian group $C_k(V)$ on Z generated by the set of the oriented k-strata V^k_j of V in which we identify the elements with opposite orientations and multiplicity in Z. With such hypotheses we define the "reduction" of ξ as the new chain $\xi_/ := (V_{/z}, z)$ obtained by restricting the support of V to its essential part: i.e. we consider only the strata A adjacent to some V_j with maximal dimension k (hence $A \subseteq \overline{V_j}$) and multiplicity $n_j \neq 0$ in Z. Explicitly $V_{/z} := \bigcup_{n_j \neq 0} \overline{V_j} = \overline{\bigcup_{n_j \neq 0} V_j}$ with the obvious induced stratification. A k-cycle $\xi = (V, z)$ is a chain whose boundary $\partial\xi$ is zero, where $\partial\xi$ is defined by the reduction $\partial\xi := (V_{k-1}, \partial z)_/$ and ∂z is given by the homological boundary operator through the natural isomorphisms ψ_k, ψ_{k-1} as (see 3.2) in the diagram

$$\partial : C_k(V) \xrightarrow{\psi_k} H_k(V_k, V_{k-1}) \xrightarrow{\partial_k} H_{k-1}(V_{k-1}, V_{k-2}) \xrightarrow{\psi_{k-1}^{-1}} C_{k-1}(V_{k-1}) \, .$$

Fix now a k-cycle $\xi = (V, z)$; we have $\partial z = 0$ and therefore $\partial_k \psi_k(z) = 0$. Now by the exactness of the pairs (V_k, V_{k-1}) and (V_{k-1}, V_{k-2}) we have that the maps i_* and j_* induced by the inclusions (below defined) are monomorphisms, and then looking at the diagram

$$\begin{tikzcd}
H_k(V_{k-1}) \arrow[r] & H_k(V_k) \arrow[r, "i_*"] \arrow[d, "I_*"'] & H_k(V_k, V_{k-1}) \arrow[r, "\partial"] \arrow[rr, bend left, "\partial_k"] & H_{k-1}(V_{k-1}) \arrow[r, "j_*"] & H_{k-1}(V_{k-1}, V_{k-2}) \\
 & H_k(X) & C_k(V_k) \arrow[u, "\psi_k"] \arrow[rr, "\partial"'] & & C_{k-1}(V_{k-1}) \arrow[u, "\psi_{k-1}"']
\end{tikzcd}$$

we get $0 = \partial_k \psi_k(z) = j_* \partial \psi_k(z)$ and $\partial \psi_k(z) = 0$. Hence $\psi_k(z) \in \operatorname{Ker} \partial = \operatorname{Im} i_*$ so it comes from a unique element $i_*^{-1}(\psi_k(z))$ whose image in $H_k(X)$, $R(\xi) = I_* i_*^{-1}(\psi_k(z))$ is called the "fundamental class" of ξ in $H_k(X)$.

Denoting by $WH_k(X)$ the quotient of the k-cycles modulo the cobordism relation, in [1] Goresky proved the following:

Theorem. – *The map $R_k : WH_k(X) \to H_k(X)$ is well-defined for every Whitney object X and is a bijection whenever X is a manifold.*

2.2 – Transversality

The "nice" morphisms for the Whitney category are the stratified maps $f : X \to Y$ i.e. those which map each stratum R of X smoothly in some stratum S_R of Y. The definitions of transversality are those one could expect: we require the usual condition on each stratum. With the above notation, from lemma 5.3. in [1] we have immediately

Theorem. – *If a stratified map $f : X \to Y$ between the objects X and Y in the manifolds M and M' (is controlled or) has a smooth extension $\overline{f} : M \to M'$, then every k-cycle $\xi = (V, z)$ of Y equipped with π-fibres*[(1)] *is cobordant to a k-cycle $\xi' = (V', z')$ with f transversal to V'.*

In particular, if we write f for the inclusion map $V \hookrightarrow X$, we have that if $\xi_1 = (V_1, z_1)$ and $\xi_2 = (V_2, z_2)$ are two k-cycles of X there exists a cycle $\xi_2' = (V_2', z_2')$ which is cobordant to ξ_2 and transversal to ξ_1.

It is known that the transverse intersection $U \cap V$ of two arbitrary subspaces U and V of X is a Whitney subspace of X; we would like to show that their union $U \cup V$ is a Whitney subspace too.

First, we remark that for two manifolds $M, N \subseteq R^q$ the following obvious property holds:

Lemma. – *If (M, N) verifies Whitney's condition B then each pair (M', N') of submanifolds N' of N and M' of M with* $\dim M' = \dim M$ *verifies the same condition. In particular this holds when M' and N' are connected components of M and N.*

Proposition. – *The union $V \cup W$ of two stratified subspaces V and W wich are transversal in X is again a Whitney subspace of X.*

Proof. – Since

$$V \cup W = (V \cap W) \sqcup (V - W) \sqcup (W - V) \quad \text{and} \quad V = \bigsqcup_{A' \subseteq V} A', \quad W = \bigsqcup_{B' \subseteq W} B'$$

(A',B' strata) we have that $V \cup W$ has a natural partition in smooth connected manifolds. Notice that $A - W$ is a submanifold of A with $\dim A - V = \dim A$, being open in A, and the same holds for $B - V$.

In order to show that the h-skeleton $(V \cup W)_h$ is closed in $V \cup W$, it is sufficient to observe that $V_h \cap W \subseteq (V \cap W)_h$ and similarly $W_h \cap V \subseteq (V \cap W)_h$ and then use the identity

$$(V \cup W)_h = (V \cap W)_h \cup (V_h - W) \cup (W_h - V) = (W \cap V)_h \cup V_h \cup W_h\,.$$

Given two strata Z, Z' of $V \cup W$ to prove Whitney's condition B, by the above lemma it is not restrictive to assume that they are of the type $A \cap B, \quad A - W, \quad B - V$ and to examine the following case:

(1) This mean that V verifies the π-fibre condition (see [1]).

1) if $Z = A \cap B$ and $Z' = A' \cap B'$ then (Z, Z') verifies W.C.B because $V \cap W$ is a Whitney object;
2) if $Z = A \cap B$ and $Z' = A' - W$ then we apply the above lemma to (A, A');
3) if $Z = A - W$, $Z' = A' \cap B'$ then $Z \cap \overline{Z'} = (A - W) \cap \overline{A' \cap B'} \subseteq (V - W) \cap W = \emptyset$.

All the other cases can be dealt with in a similar way.

2.3 – The transverse sum in $WH_k(X)$

If X is a manifold the bijection $R : WH_k(X) \to H_k(X)$ induces in an obvious way a group operation in $WH_k(X)$, but we want to give it a precise geometrical meaning.

First we observe that if V and W are two k-subspaces transverse in X, then each k-orientation $z = \sum n_j V_j$ of V can be interpreted as a k-orientation of the union $V \cup W$ substituting each stratum V_j with the connected components of the manifold in which it is subdivided by the stratification of $V \cup W$. In such a way we have a natural group inclusion $\beta : C_k(V) \to C_k(V \cup W)$ by which the following diagram is commutative

$$\begin{array}{ccc} C_k(V) & \xrightarrow{\psi_k} & H_k(V_k, V_{k-1}) \\ \beta\Big\downarrow & & \Big\downarrow i_* \\ C_k(V \cup W) & \xrightarrow{\chi_k} & H_k(V \cup W, (V \cup W)_{k-1}) \,. \end{array}$$

Therefore, given two chains $\xi_V = (V, z_V)$ and $\xi_W = (W, z_W)$, embedding their orientations in the same group $C_k(V \cup W)$, we can sum them obtaining the chain

$$\xi_V + \xi_W := \big(V \cup W, \beta_V(z_V) + \beta_W(z_W)\big)$$

(of X) which is called the "*transversal sum*" of ξ_V and ξ_W.

Proposition. – $\partial(\xi_V + \xi_W) = \partial\xi_V + \partial\xi_W$. *In particular the transverse sum of two k-cycles is again a k-cycle.*

Proof. – Considering the k-subspace $V = V_k$ and the $(k-1)$-skeleton V_{k-1}, we find the commutative diagrams (1) and (2).

On the other hand the $(k-1)$-strata of $V_{k-1} \cup W_{k-1}$ are also $(k-1)$-strata in $(V \cup W)_{k-1}$, producing the inclusion homomorphism J which corresponds in singular homology to the homomorphism γ_* induced by the inclusion of pairs γ.

Therefore, we have the commutative diagram

$$\begin{tikzcd}
C_k(V) \arrow[r, "\psi_k"] \arrow[rrr, bend left, "\partial_V"] \arrow[dd, "\beta_{V_k}"'] \arrow[ddr, phantom, "(1)"] & H_k(V_k, V_{k-1}) \arrow[r, "\partial_k"] \arrow[dd, "\alpha_*"'] & H_{k-1}(V_{k-1}, V_{k-2}) \arrow[r, "\psi_{k-1}^{-1}"] \arrow[d, "\alpha_{/*}"'] \arrow[dr, phantom, "(2)"] & C_{k-1}(V_{k-1}) \arrow[d, "\beta_{V_{k-1}}"] \\
 & & H_{k-1}(V_{k-1}\cup W_{k-1}), (V_{k-1}\cup W_{k-1})_{k-2}) \arrow[r, "\phi_{k-1}^{-1}"] \arrow[d, "\gamma_*"'] & C_{k-1}(V_{k-1}\cup W_{k-1}) \arrow[d, "J"] \\
C_k(V\cup W) \arrow[r, "\chi_k"] \arrow[rrr, bend right, "\partial_{V\cup W}"'] & H_k(V\cup W, (V\cup W)_{k-1}) \arrow[r, "\partial_k"] & H_{k-1}((V\cup W)_{k-1}, (V\cup W)_{k-2}) \arrow[r, "\chi_{k-1}^{-1}"] & C_{k-1}((V\cup W)_{k-1})
\end{tikzcd}$$

and so the relation

$$\partial_{V\cup W}\beta_{V_k}(z_V) = J\beta_{V_{k-1}}\partial_V(z_V)\,.$$

Likewise we find for W

$$\partial_{V\cup W}\beta_{W_k}(z_W) = J\beta_{W_{k-1}}\partial_W(z_W)$$

and summing in $C_{k-1}\big((V\cup W)_{k-1}\big)$ we have the equality

$$\partial_{V\cup W}\Big(\beta_{V_k}(z_V) + \beta_{W_k}(z_W)\Big) = J\Big(\beta_{V_{k-1}}\partial_V(z_V) + \beta_{W_{k-1}}\partial_W(z_W)\Big)\,.$$

Now, it is easy to get, by the definition of reduction of a geometric chain, the following relations:

a) $(V_{k-1}\bigcup W_{k-1})_{/\beta_{V_{k-1}}(z_1)+\beta_{W_{k-1}}(z_2)} = V_{k-1/_{z_1}}\bigcup W_{k-1/_{z_2}}$

b) $(V\bigcup W)_{k-1_{/J(z)}} = (V_{k-1}\bigcup W_{k-1})_{/z}$

So we are able to conclude that

$$\begin{aligned}
&\partial(\xi_V+\xi_W) = \big(V\cup W, \beta_V(z_V)+\beta_W(z_W)\big)_{/} = \big((V\cup W)_{k-1/J(\ldots)}, J(\ldots)\big) = \\
&= \big(V_{k-1}\cup W_{k-1/..}, ..\big) = \big(V_{k-1/\partial z_V}\cup W_{k-1/\partial z_W}, \beta_{V_{k-1}}\partial(z_V)+\beta_{W_{k-1}}\partial(z_W)\big) = \\
&= \big(V_{k-1/\partial z_V}, \partial z_V\big) + \big(W_{k-1/\partial z_W}, \partial z_W\big) = \partial\xi_V + \partial\xi_W\,.
\end{aligned}$$

THEOREM. – *Let X^n be a smooth manifold without boundary. For each pair ξ_{V_1}, ξ_{V_2} of k-cycles of X define*

$$[\xi_{V_1}] + [\xi_{V_2}] := [\xi_{V_1} + \xi_{V'_2}]$$

where $\xi_{V'_2}$ is a k-cycle of X cobordant to ξ_{V_2} and transverse to ξ_{V_1}. Then we have a well defined group operation in $WH_k(X)$ with respect to which the Goresky representation $R_k : WH_k(X) \to H_k(X)$ becomes an isomorphism of abelian groups.

Proof. – Let $\xi_{V_1} \equiv \xi_{W_1}$, $\xi_{V_2} \equiv \xi_{W_2}$ be k-cycles of X which we can suppose to be reduced, and let furthermore $\xi_{V'_2}$, $\xi_{W'_2}$ be two cycles of X such that:

$\xi_{V'_2}$ is cobordant to ξ_{V_2} and transversal to ξ_{V_1}

$\xi_{W'_2}$ is cobordant to ξ_{W_2} and transversal to ξ_{W_1}.

We remark that all subspaces of a manifold X are obviously equipped with π-fibres and so such cycles $\xi_{V'_2}$ and $\xi_{W'_2}$ exist by the transversality lemma.

If we assume again these to be both reduced we have to show that:

$$\xi_{V_1} + \xi_{V'_2} \equiv \xi_{W_1} + \xi_{W'_2}$$

i.e. they are cobordant.

First we observe that the space $X_2 = X \times [0,1]$ is an $(n+1)$-Whitney object ($\dim X = n$) with n-skeleton $(X_2)_n = X \times 0 \cup X \times 1$ in which

$$\xi_{W'_2} \times 1 - \xi_{V'_2} \times 0 \quad \text{is transverse to} \quad \xi_{W_1} \times 1 - \xi_{V_1} \times 0\,.$$

Since $\xi_{V_1} \equiv \xi_{W_2}$, there is a $(k+1)$-chain θ_{U_1} in $X \times [0,1]$ and any $\varepsilon > 0$ such that:

$a_1)$ $U_1 \cap (X \times [0,\varepsilon[) = V_1 \times [0,\varepsilon[; U_1 \cap (X \times]1-\varepsilon,1]) = W_1 \times]1-\varepsilon,1]$

$b_1)$ $\partial\theta_{U_1} = \xi_{W_1} \times 1 - \xi_{V_1} \times 0$

and similary for $\xi_{V'_2} \equiv \xi_{W'_2}$ we can find a $(k+1)$-chain $\theta_{U'_2}$ such that:

$a'_2)$ $U'_2 \cap (X \times [0,\varepsilon[) = V'_2 \times [0,\varepsilon[\quad ; \quad U'_2 \cap (X \times]1-\varepsilon,1] = W'_2 \times]1-\varepsilon,1]$

$b'_2)$ $\partial\theta_{U'_2} = \xi_{W'_2} \times 1 - \xi_{V'_2} \times 0.$

Thus by $b_1)$ and $b'_2)$ we have that the part $\partial\theta_{U'_2}$ of $\theta_{U'_2}$ contained in $(X_2)_n$ is transverse to $\partial\theta_{U_1}$ (of θ_{U_1}) which is also in $(X_2)_n$.

By $a_1)$ and $a'_2)$ such transversality holds in the open neighbourhood $U = X \times ([0,\varepsilon[\cup]1-\varepsilon,1])$ of $(X_2)_n$. Now, since $\theta_{U'_2}$ is equipped with π-fibres in X_2, in the same way as in the inductive passage of the proof of the transversality lemma (see [1] 5.3.), we can find a $(k+1)$-chain ζ in X_2 coinciding with $\theta_{U_2'}$ in U and transverse to θ_{U_1} in all of X_2.

So the transverse sum chain $\theta_{U_1}+\zeta$ exists and (since $|\zeta|\cap U = |\theta_{U_2'}|\cap U$) by a_1) and a_2') its support satisfies

a) $|\theta_{U_1}+\zeta|\cap(X\times[0,\varepsilon[) = (|\theta_{U_1}|\cup|\zeta|)\cap(X\times[0,\varepsilon[) = (V_1\cup V_2')\cap(X\times[0,\varepsilon[) = (V_1\cup V_2')\times[0,\varepsilon[$;

$$|\theta_{U_1}+\zeta|\cap(X\times]1-\varepsilon,1]) = \ldots \text{similarly} \ldots = (W_1\cup W_2')\times]1-\varepsilon,1])$$

and by the previous proposition

b) $\partial(\theta_{U_1}+\zeta) = \partial\theta_{U_1}+\partial\zeta = \partial\theta_{U_1}+\partial\theta_{U_2'} = (\xi_{W_1}\times 1-\xi_{V_1}\times 0)+(\xi_{W_2'}\times 1-\xi_{V_2'}\times 0) = (\xi_{W_1}+\xi_{W_2'})\times 1-(\xi_{V_1}+\xi_{V_2'})\times 0$.

Thus $\theta_{U_1}+\zeta$ is the required cobordism, and $\xi_{V_1}+\xi_{V_2'}$ is a cycle, again by the previous proposition.

About the second statement, it is sufficient to prove that

$$R([\xi_1+\xi_2]) = R([\xi_1]) + R([\xi_2]).$$

Fix two cycles $\xi_1=(V_1,z_1)$, $\xi_2=(V_2,z_2)$ transverse in X, and denote by $i_1,i_2,i,\ I_1,I_2,I,\alpha_1,\alpha_2$ the inclusions which induce (in homology) the homomorphisms appearing in the following diagrams

$$\begin{array}{ccc} C_k(V_1) & \overset{\psi_k}{\to} & H_k(V_1,|V_1|_{k-1}) \\ \beta_{V_1}\downarrow & & \downarrow\alpha_{1*} \\ C_k(V_1\cup V_2) & \overset{\chi_k}{\to} & H_k(V_1\cup V_2,|V_1\cup V_2|_{k-1}) \end{array} \qquad \begin{array}{ccc} C_k(V_2) & \overset{\phi_k}{\to} & H_k(V_2,|V_2|_{k-1}) \\ \beta_{V_2}\downarrow & & \downarrow\alpha_{2*} \\ C_k(V_1\cup V_2) & \overset{\chi_k}{\to} & H_k(V_1\cup V_2,|V_1\cup V_2|_{k-1}) \end{array}$$

and in the diagrams

$$\begin{array}{ccc} H_k(V_1,|V_1|_{k-1}) & \overset{i_{1*}}{\leftarrow} & H_k(V_1) \\ \psi_k\uparrow & \nearrow & \downarrow I_{1*} \\ C_k(V_1) & & H_k(X) \end{array} \qquad \begin{array}{ccc} H_k(V_2,|V_2|_{k-1}) & \overset{i_{2*}}{\leftarrow} & H_k(V_2) \\ \phi_k\uparrow & \nearrow & \downarrow I_{2*} \\ C_k(V_2) & & H_k(X) \end{array}$$

$$\begin{array}{ccc} H_k(V_1\cup V_2,|V_1\cup V_2|_{k-1}) & \overset{i_*}{\leftarrow} & H_k(V_1\cup V_2) \\ \chi_k\uparrow & \nearrow & \downarrow I_* \\ C_k(V_1\cup V_2) & & H_k(X) \end{array}$$

We then get the following equalities

a) $R(\xi_1) = I_{1*}i_{1*}^{-1}\psi_k(z_1)$; $R(\xi_2) = I_{2*}i_{2*}^{-1}\phi_k(z_2)$

b) $R(\xi_1+\xi_2) = I_* i_*^{-1} \chi_k(\beta_{V_1}(z_1)+\beta_{V_2}(z_2))$.

Now the inclusions of pairs

$$I_{12} : (V_1, \emptyset) \to (V_1 \cup V_2, \emptyset) \quad , \quad I_{21} : (V_2, \emptyset) \to (V_1 \cup V_2, \emptyset)$$

evidently verify the relations:

a') $\alpha_1 \circ i_1 = i \circ I_{12} \quad , \quad \alpha_2 \circ i_2 = i \circ I_{21}$;

b') $I_1 = I \circ I_{12} \quad , \quad I_2 = I \circ I_{21}$

so we find that

$$\chi_k(\beta_{V_1}(z_1)+\beta_{V_2}(z_2)) = \chi_k \beta_{V_1}(z_1) + \chi_k \beta_{V_2}(z_2) = \alpha_{1*}\psi_k(z_1) + \alpha_{2*}\phi_k(z_2) =$$
$$= i_* I_{12*} i_{1*}^{-1} \psi_k(z_1) + i_* I_{21*} i_{2*}^{-1} \phi_k(z_2) = i_*\big(I_{12*} i_{1*}^{-1}\psi_k(z_1) + I_{21*} i_{2*}^{-1}\phi_k(z_2)\big).$$

Hence

$$i_*^{-1}\chi_k(\beta_{V_1}(z_1)+\beta_{V_2}(z_2)) = I_{12*} i_{1*}^{-1}\psi_k(z_1) + I_{21*} i_{2*}^{-1}\phi_k(z_2)$$

and finally we conclude that

$$\begin{aligned} R(\xi_1+\xi_2) &= I_* i_*^{-1}\big(\chi_k(\beta_{V_1}(z_1)+\beta_{V_2}(z_2))\big) = \\ &= I_*\big(I_{12*}(i_{1*}^{-1}\psi_k(z_1)) + I_{21*}(i_{2*}^{-1}\phi_k(z_2))\big) = \\ &= I_{1*} i_{1*}^{-1}\psi_k(z_1) + I_{2*} i_{2*}^{-1}\phi_k(z_2) = R(\xi_1) + R(\xi_2). \end{aligned}$$

Corollary. – *The sum of two cycles $\xi_1 = (V, z_1)$ and $\xi_2 = (V, z_2)$ having the same support V is given by the sum of their orientations: i.e. $[\xi_1 + \xi_2] = [(V, z_1 + z_2)]$. The zero element of $WH_k(X)$ is the null cycle $0 = (\emptyset, 0)$. The opposite class of each $\xi = (V, z)$ is given by inverting the orientation in $C_k(V)$: i.e. $-[\xi] = [(V, -z)]$.*

Proof. – Considering the cycle $\xi = (V, z_1 + z_2)$ we find

$$\begin{aligned} R([\xi_1]+[\xi_2]) &= R([\xi_1]) + R([\xi_2]) = I_* i_*^{-1}\psi_k(z_1) + I_* i_*^{-1}\psi_k(z_2) = \\ &= I_* i_*^{-1}\psi_k(z_1+z_2) = R([\xi]) \end{aligned}$$

and we use the bijectivity of R.

The others assertions follow similarly.

2.4 – Homology with coefficients in an abelian group

Fix a Whitney object X and an abelian group G.

For every k-subspace V of X we can define a *k-orientation of V with coefficients in G* to be a formal sum of the type $z = \sum_j g_j V_j$ where $j \in J_k, g_j \in G$ and $g_j \neq 0$ for a finite number of j, for which V_j is G-oriented.

We thus construct (instead of the $C_k(V)$) the new group

$$C_k(V;G) = \left\{ \sum g_j V_j \mid \ldots \text{ and } V_j \text{ is } G\text{-oriented} \right\}.$$

In this way Whitney chains with coefficients in G are defined as pairs $\xi = (V, z)$ where $z \in C_k(V;G)$. It is now important to note that in this theory we still have the isomorphism $\psi_k : C_k(V;G) \to H_k(V, V_{k-1};G)$ (see 2.3.).

Then it is also clear how we must define the boundary of an orientation $z \in C_k(V;G)$ of a chain $\xi = (V, z)$, and then, with obvious meaning of the symbols, the set $WH_k(X;G)$.

Rereading the essential steps of the above proof we easly find that all the results of § 2 extend to $WH_k(X;G)$. We then have:

THEOREM. – *If X is a smooth manifold without boundary, then the representation map $R : WH_k(X;G) \to H_k(X;G)$ is a group isomorphism.*

3. Whitney cohomology

In the following we show in what way the obtained results can be extended to the cohomological case for every Whitney space X (not necessarily a manifold). It will be crucial that the cocycles of X are equipped with π-fibres by definition, and thus the transversality lemma holds for them.

3.1 – Costrata and coorientability

Let V be a π-fibre of X. We call k-costratum of V, every connected component D^s_k $(s \in \Gamma_k)$ of the union $\bigcup_{codim(A)=k} A$ (A stratum embedded in X with codimension k). V is called a k-π-fibre of X if it has only costrata (and hence strata) with dimension at least k (k fixed). In such a way the p-coskeleton defined by $V^p := \bigcup_{k' \geq p} \bigcup_s D^s_{k'}$ is the support of a p-π-fibre V^p of X, and we have

$$\emptyset = V^{n+1} \subseteq V^n \subseteq \ldots \subseteq V^k = \ldots = V^0 = V\,.$$

A k-costratum D has each of its strata A lying in some stratum S_A of X with codimension k; then the normal bundle $E_A = \perp (A, S_A)$ is defined with vector space fibre $E_p = \perp_p (A, S_A)$, $\forall p \in A$, having dimension k independent of the dimension of A; i.e.:

$$\dim E_p = \dim \perp_p (A, S_A) = \operatorname{codim} A = k.$$

These bundles E_A fit together by the p-π-fibre condition (see [1]) determining a global bundle $E_D = \bigcup_{A \subseteq D} E_A$ over the costratum D with fibres isomorphic to R^k.

Writing $\dot{E}_D = E_D - D$ $(D = 0-\text{section})$ we have a fibre-bundle pair $(E_D, \dot{E}_D)$ over D with fibres $(E_p, \dot{E}_p) \cong (R^k, R^k - 0)$. By homotopy we may think of this as a sphere-bundle and carry over the definitions and properties concerning the orientability. Thus D is called *$G-$coorientable* if $(E_D, \dot{E}_D)$ is a G-orientable sphere bundle pair. Then G-coorientability is characterized by the cohomology $H^k(E_D, \dot{E}_D; G)$.

We fix [10] chap.5 sec.7 for reference and various notations.

Proposition 1. – $H^k(E_D, \dot{E}_D; G) \cong \begin{cases} G & \textit{if } D \textit{ is } G\textit{-coorientable}; \\ 0 & \textit{otherwise.} \end{cases}$

Proof. – If D is G-coorientable, then by Thom isomorphism we have

$$H^k(E_D, \dot{E}_D; G) \cong H^0(D, G) \cong G.$$

If D is not G-coorientable there exists a covering $\Omega = \{V\}_V$ of D (see [10] cor. 17) for which there is a compatible family $u_\Omega = \{u_V\}_{V \in \Omega}$with every $u_V \in H^k(E_V, \dot{E}_V; G)$. Therefore the module $H^k(\{p^{-1}(V)\}, E_D; G)$ formed by all the compatible collections over Ω must be null. Then by isomorphism ([10] lemma 6) we find

$$H^k(E_D, \dot{E}_D; G) \cong H^k(\{p^{-1}(V)\}, E_D; G) \cong 0.$$

Considering now the free abelian group over the set of all G-cooriented k-costrata D of V, i.e.

$$C^k(V^k; G) := \left\{ \sum_{s \in \Gamma_k} g_s D_s^k \mid \ldots \text{and } D_s^k \text{ is } G\text{-cooriented} \right\}$$

obtained by the usual identification $g \cdot D = (-g) \cdot (-D)$ we have:

Proposition 2. – *There is a natural isomorphism*

$$\psi^k : C^k(V^k; G) \longrightarrow H^k(X - V^{k+1}, X - V^k; G)\,.$$

Proof. – See 3.3.

Every element of $C^k(V;G)$ will be called a *k-coorientation of V with coefficients in G.*

In the next section all coefficients are in a fixed abelian group G.

3.2 – The isomorphism $R^k : WH^k(X;G) \to H^k(X;G)$

A k-cochain in X is a pair $\theta = (V, c)$ where V is a k-π-fibre of X and $c = \sum\limits_j g_j D_j$ is a k-coorientation of V. The reduction $\theta_/ = (V_{/c}, c)$ of θ is obtained by restricting the support of V to costrata D_j with multiplicity $g_j \neq 0$. θ is called a k-cocycle if $\delta\theta = 0$, where the latter is the reduction $\delta\theta = (V^{k+1}, \delta c)_/$ and δc is defined through the cohomological boundary by

$$\delta : C^k(V^k) \overset{\psi^k}{\to} H^k(X-V^{k+1}, X-V^k) \overset{\delta_k}{\to} H^{k+1}(X-V^{k+2}, X-V^{k+1}) \overset{(\psi^{k+1})^{-1}}{\longrightarrow} C^{k+1}(V^{k+1}).$$

If $\theta = (V, c)$ is a cocycle, using the exact sequence of the triads

$$X - V^k \subseteq X - V^{k+1} \subseteq X \quad , \quad X - V^{k+1} \subseteq X - V^{k+2} \subseteq X$$

in the diagram

$$\begin{tikzcd}
 & & & H^{k+1}(X, X-V^{k+2}) \arrow[d, "h^*"] \\
H^k(X, X-V^{k+1}) \arrow[r, "j^*"] & H^k(X, X-V^k) \arrow[r, "i^*"] \arrow[d, "I^*"] & H^k(X-V^{k+1}, X-V^k) \arrow[r, "\delta"] \arrow[dr, "\delta_k"] & H^{k+1}(X, X-V^{k+1}) \arrow[d, "H^*"] \\
 & H^k(X) & C^k(V) \arrow[u, "\psi^k"] & H^{k+1}(X-V^{k+2}, X-V^{k+1})
\end{tikzcd}$$

we have $\delta c = 0$, and hence $\delta_k \psi^k(c) = \psi^{k+1}\delta(c) = 0$ so $\psi^k(c) \in Ker\,\delta_k = Ker\,\delta = Im\,I^*$ (see 4.5. [1]). Thus a unique element ${i^*}^{-1}\psi^k(c)$ is determined in $H^k(X, X-V^k)$ whose image $R^k(\theta) = I^* {i^*}^{-1} \psi^k(c)$ in $H^k(X)$ is called the fundamental class of θ in $H^k(X)$.

Denoting by $WH^k(X)$ the quotient of the cocycles modulo the cobordism relation, in [1] Goresky proved the following:

THEOREM. – *For every Whitney space X the map $R^k : WH^k(X) \to H^k(X)$ is well defined and bijective.*

3.3 – (About the) duality between the maps ψ_k and ψ^k

The homological ψ_k. If V is a k-cycle of $X \in M$ the isomorphism $\psi_k : C_k(V) \overset{\cong}{\to} H_k(V_k, V_{k-1})$ is defined as follows. Fix a system of control data $\{(T_A, \pi_A, \rho_A)\}_A$ of V; $\forall \rho > O$ we denote by $T_\rho A$ the tubular neighbourhood of A in V (i.e. $T_\rho A = T_A(\rho) \cap V$) and by $T_\rho(V_{k-1}) = \bigcup_{dimA \leq k-1} T_\rho A$. The latter retracts over V_{k-1} and thus we have for some $\varepsilon > 0$ $\quad H_k(V_k, V_{k-1}) \cong H_k(V_k, T_{2\varepsilon}(V_{k-1}))$. Now by excision of $T_\varepsilon(V_{k-1})$ we get:

$$\begin{cases} V_k - T_\varepsilon(V_{k-1}) = (V - V_{k-1}) - T_\varepsilon(V_{k-1}) = \displaystyle\bigcup_{S=k-stratum} S_\varepsilon \\ T_{2\varepsilon}(V_{k-1}) - T_\varepsilon(V_{k-1}) = \displaystyle\bigcup_{S=k-stratum} (S_\varepsilon - S_{2\varepsilon}) \end{cases}$$

where $S_\varepsilon = \{p \in S \mid \rho_A(p) \geq \varepsilon, \quad \forall A \quad \text{stratum of} \quad V_{k-1}\}$ is a k-manifold (with corners) and $S_\varepsilon - S_{2\varepsilon}$ retracts over ∂S_ε. Thus we have by excision and homotopy (see [10] cap.6 sec.3)

$$H_k(V_k, V_{k-1}; G) \cong H_k(V_k, T_{2\varepsilon} V_{k-1}) \cong H_k(\bigsqcup S_\varepsilon, \bigsqcup \partial S_\varepsilon) \cong \bigoplus_{S_\varepsilon} H_k(S_\varepsilon, \partial S_\varepsilon) \cong$$

$$\cong \bigoplus_{\substack{S=k-stratum \\ G-orientable}} G_S = C_k(V_k).$$

The cohomological ψ^k. If V is a k-cocycle of X, the isomorphism ψ^k is defined as follows. The tubular neighbourhoods of the costrata D of V are given by

$$T_D = \bigcup_{A \subseteq D} T_A \quad , \qquad T(V^k - V^{k+1}) = \bigcup_{D=k-costratum} T_D.$$

Therefore excising $X - T(V^k - V^{k+1})$ from the pair $(X - V^{k+1}, X - V^k)$ we get

$$\begin{cases} X - V^{k+1} - \big(X - T(V^k - V^{k+1})\big) = T(V^k - V^{k+1}) = \displaystyle\bigcup_{D=k-costratum} T_D \\ X - V^k - \big(X - T(V^k - V^{k+1})\big) = T(V^k - V^{k+1}) - (V^k - V^{k+1}) = \displaystyle\bigcup_{D=k-costratum} T_D - D. \end{cases}$$

Thus

$$H^k(X - V^{k+1}, X - V^k) \cong H^k(\bigcup T_D, \bigcup T_D - D) \cong \bigoplus_{D=k-costratum} H^k(T_D, T_D - D)$$

but since $T_D \cong E_D$ and by proposition 2.1., the latter is also isomorphic to

$$\bigoplus_{D=k-costratum} H^k(E_D, \dot{E}_D) \quad \cong \bigoplus_{\substack{D=k-costratum \\ G-coorientable}} G_D = C^k(V^k; G).$$

It is important to notice that the isomorphisms ψ_k and ψ^k are essentially given by homomorphisms induced by inclusions.

Finally we can now recall simultaneously the definitions of two representation maps, the homological and the cohomological

$$R_k : WH_k(X) \to H_k(X) \quad ; \qquad R^k : WH^k(X) \to H^k(X)$$

immediately noticing that if $\xi = (V, z)$ is a k-cycle or if $\theta = (V, c)$ is a k-cocycle, with the homomorphisms in the following diagrams

$$\begin{array}{ccc} H_k(V_k, V_{k-1}) & \overset{i_*}{\longleftarrow} & H_k(V_k) \\ \psi_k \uparrow & \nearrow & \downarrow I_* \\ C_k(V) & & H_k(X) \end{array} \qquad \begin{array}{ccc} H^k(X - V^{k+1}, X - V^k) & \overset{i^*}{\longleftarrow} & H^k(X, X - V^k) \\ \psi^k \uparrow & \nearrow & \downarrow I^* \\ C^k(V) & & H^k(X) \end{array}$$

by definition we have

$$R_k(\xi) = I_* i_*^{-1} \psi_k(z) ; \qquad R^k(\theta) = I^* i^{*-1} \psi^k(c) .$$

Then we find a situation of complete parallel between the homological case and the cohomological case. Therefore we can develop a detailed analysis of § 2 in a cohomological version carrying out in the propositions and (in the) proofs the following substitutions:

$k - \pi - fibre\,,\ k - cocycle$	instead of	$k - subspace\,,\ k - cycle$
$k - costratum\,,\ k - coskeleton$		$k - stratum\,,\ k - skeleton$
$k - coorientation\,,\ C^k(V^k)$		$k - orientation\,,\ C_k(V_k)$
$\psi^k :\ C^k(V^k)\ \cong\ H^k(X - V^{k+1}, X - V^k)$		$\psi_k :\ C_k(V)\ \cong\ H_k(V_k, V_{k-1})$
$\delta :\ C^k(V^k)\ \to\ C^{k+1}(V^{k+1})$		$\partial :\ C_k(V_k)\ \to\ C_{k-1}(V_{k-1})$

The final result is that the whole section "1.2. The transverse sum in $WH_k(X)$" may be developed in a formally analogous way considering the cohomology $WH^k(X)$ of an arbitrary space X rather than the homology of a manifold X.

Remembering that in the cohomological case, given two cocycles θ_{V_1} and θ_{V_2} by the transversality lemma there is (always) a cocycle $\theta_{V_2'}$ which is cobordant to θ_{V_2} and transverse to θ_{V_1}, we have the following:

Theorem. – *For every Whitney object X the representation map $R^k : WH^k(X) \to H^k(X)$ is a group isomorphism.*

3.4 – Induced homomorphisms

If $f : X_1 \to X_2$ is a controlled map by the transversality lemma each element of $WH^k(X_2)$ can be represented by a k-cocycle $\theta = (V, c)$ transversal to f; thus we find a partition (in strata) of $f^{-1}(V) = \cup_{A \subseteq V} f^{-1}(A)$ in such a way that this is again a k-π-fibre of X_1. Therefore the k-coorientation $c = \sum g_j D_j$ induces a k-coorientation $f^{-1}(c) = \sum g_j f^{-1}(D_j)$ of $f^{-1}(V)$ defining a function f^* which maps every cocycle of X_2 onto its transversal preimage in X_1.

Proposition. – *The map $f^* : WH^k(X_2) \to WH^k(X_1)$ is a group homomorphism.*

Proof. – In [1] it is proved that f^* is well-defined and that the diagram

$$\begin{array}{ccc} WH^k(X_2) & \stackrel{f^*}{\to} & WH^k(X_1) \\ R_{X_2} \downarrow & & \downarrow R_{X_1} \\ H^k(X_2) & \stackrel{f^*}{\to} & H^k(X_1) \end{array}$$

is commutative.

Thus $f^* = R^{-1} \circ f^* \circ R$ being a composition of homomorphisms is itself a homomorphism.

Preliminary remarks: For the following § 4 and § 5 we consider the coefficients in the field Z_2. Then we have a convenient notational simplification since the coorientation c of each (reduced) k-cocycle $\theta = (V, c)$ is uniquely determined by the support V so we can, and shall, understand c and denote θ simply by its support V.

4. The Steenrod squares

We will give the definition of the maps $Sq^\alpha : WH^k(X) \to WH^{k+\alpha}(X)$ through a geometric construction substantially consisting of two steps:

a) first we introduce a function $\varphi : WH^k(X) \to WH^{2k}(X^2 \times S^h/Z_2)$ defined by $V \mapsto V^2 \times S^h/Z_2$ where the space $X^2 \times S^h/Z_2$ and its cocycle $V^2 \times S^h/Z_2$ are factorized modulo the Z_2-action $A : X^2 \times S^h \to X^2 \times S^h$, $A(x,y,t) = (y,x,-t)$.

b) secondly, the diagonal map $\Delta : X \times P^h \to X^2 \times S^h/Z_2, \Delta(x,[t]) = [x,x,t]$ induces the homomorphism $\Delta^* : WH^{2k}(X^2 \times S^h/Z_2) \to WH^{2k}(X \times P^h)$, in such a way that, by the Künneth formula

$$\Delta^*(V^2 \times S^h/Z_2) \in WH^{2k}(X \times P^h) \cong \sum_{i+j=2k} WH^i(X) \bigotimes WH^j(P^h)$$

and its "component" in $WH^{k+\alpha}(X)$ multiplied by the coefficient $b_{k-\alpha}$ of the Euler class $e^{k-\alpha}$ will define the element $Sq^\alpha(V)$.

In the whole process $h \in N$ can be an arbitrary natural number provided $h \geq k - \alpha$; furthermore it will automatically be $Sq^\alpha = 0$ if $\alpha > k$.

4.1 – The map $\varphi : WH^k(X) \to WH^{2k}(X^2 \times S^h/Z_2)$

Given a Whitney space X we can assume that it has only boundaryless strata (possibly considering the boundaries as lower dimensional strata) and because it is not restrictive for $WH^k(X)$ we consider only such spaces X. In this way the product $X^2 \times S^h$ is again a Whitney space. Similarly if V is a k-cocycle of X we find a cobordant $2k$-cocycle $V^2 \times S^h$ of $X^2 \times S^h$, and these can be factorized defining the new objects $X^2 \times S^h/Z_2 \supseteq V^2 \times S^h/Z_2$ which are again a Whitney space and one of its $2k$-cocycles.

Proposition 1. – *The map φ is well defined.*

We state first two lemmas.

Lemma 1. – *If $\zeta : V \equiv V'$ is a cobordism, then we can find a cobordism $\theta : V \equiv V'$ such that θ^2 is transverse to the diagonal map $\beta : X^2 \times R \to (X \times R)^2$, $\beta(x,y,t) = (x,t,y,t)$ out of an isolated set.*

Proof. – We observe that β is already transversal to ζ^2 over the set $(X \times ([0,\varepsilon[\cup]1-\varepsilon,1]))^2$. In fact

$$\beta^{-1}(X \times [0,\varepsilon[\times X \times]1-\varepsilon,1]) = \emptyset = \beta^{-1}(X \times]1-\varepsilon,1] \times X \times [0,\varepsilon[)$$

and moreover the restriction of β to the subset $X^2 \times ([0,\varepsilon[\cup]1-\varepsilon,1])$ is clearly transverse to $(V \times [0,\varepsilon[)^2$ and to $(V' \times]1-\varepsilon,1])^2$.

Let us write $X_2 = X \times R$. Then we can construct θ inductively by a substantial repetition of the proof of the transversality lemma with the following tricks:

1) the deformation must be carried out not "moving" ζ in the neighbourhood $X \times ([0,\varepsilon[\cup]1-\varepsilon,1])$;

2) the strata A of θ must verify the condition $pr_2 T_{(x,t)} A \neq 0$, outside an isolated subset of A.

Observing that $pr_2 T_{(x,t)} A = 0$ if and only if (x,t) is a critical point for the map $pr_2 : A \to R$ and that we can obviously choose as an induction hypothesis $\theta_0 = \zeta$, we conclude because transversality follows by condition 2) and cobordism by condition 1).

Lemma 2. – *If $V \equiv V'$ are cobordant in X, then $V^2 \equiv V'^2$ are Z_2-equivariantly cobordant in X^2 with respect to the action $A : X^2 \times R \to X^2 \times R$, $A(x,y,t) = (y,x,t)$.*

Proof. – If $\theta : V \equiv V'$ is (a cobordism) as in the previous lemma, then taking its "critical points" as the 0-dimensional strata we have that β is transverse to θ^2 and $\beta^{-1}(\theta^2) : V^2 \equiv V'^2$ is the required cobordism.

To check this we only need to observe that $A\beta^{-1}(V_1 \times V_2) = \beta^{-1}(V_2 \times V_1)$ for each V_1, V_2 strata of θ.

Proof of Prop. 1. – Let $V \equiv V'$ in X and $\theta : V^2 \equiv V'^2$ the cobordism of the previous lemma. Inverting the last two coordinates we get the isomorphism $X^2 \times R \times S^h \cong X^2 \times S^h \times R$.

The cochain $\theta \times S^h \subseteq X^2 \times R \times S^h$ defines a cobordism $P : V^2 \times S^h \equiv V'^2 \times S^h$ which is Z_2-equivariant with respect to A. Thus $P/Z_2 : V^2 \times S^h/Z_2 \equiv V'^2 \times S^h/Z_2$.

We introduce now a property whose usefulness will be clear in the following.

Proposition 2. – *The cross product of two classes of Whitney cocycles is represented by the cartesian product of representative cocycles: i.e. $[V] \times [V'] = [V \times V']$.*

Proof. – The projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ verify the hypotheses of the transversality lemma, thus we can suppose p_1 transverse to V and p_2 transverse to V'. Then, by proposition 6.2. in [1], passing through the cup product in singular cohomology we find

$$R(V) \times R(V') = p_1^* R(V) \cup p_2^* R(V') = Rp_1^*(V) \cup Rp_2^*(V') = \\ = R(p_1^{-1}(V) \cap p_2^{-1}(V')) = R(V \times V').$$

4.2 – The refinement isomorphism

If X' is a Whitney refinement of X, then, by transversality to $I = id : X' \to X$ the homomorphism $I^* : WH^k(X) \to WH^k(X')$ is defined and because of $I^* = R^{-1} \circ id^* \circ R$, it is an isomorphism. Here we prove that its inverse J re-interprets the cocycles of X' as cocycles of X.

Proposition 3. – *The isomorphism J inverse of I^* is defined by*

$$J : [V]_{X'} \in WH^k(X') \to [V]_X \in WH^k(X)\,.$$

Proof. – If $V \neq 0$ is a k-cocycle in X' each of its (strata and hence) costrata has codimension at least k in X' and hence also in X. Moreover $V \neq 0$ has at least a k-costratum D. The π-fibre condition assures that the codimension of V does not change passing from the stratum S to any $S' \geq S$ in X. Thus D intersects the strata of highest dimension of X' (and of X!) and has codimension k also in X. Similarly each cobordism in $X' \times R$ is again a cobordism in $X \times R$. Therefore we deduce that J exists and is well defined.

Finally, for every cocycle V transverse to X' we have

$$J \circ I^*([V]_X) = J([I^{-1}(V)]_{X'}) = J([V \cap X']_{X'}) = [V \cap X']_X = [V]_X$$

the last identity holding with an obvious cobordism. Thus $J \circ I^* = id$ and the assertion follows by the bijectivity of I^*.

Using the transversality lemma we may also verify directly that $I^* \circ J = id$.

4.3 – Definition of Sq^{α}

Through the isomorphism $R : WH^k \cong H^k$ we can suppose that for the groups $WH^k(X)$ the following concepts are known:

a) for every $a \in WH^k(X)$ the cup product $a^j = a \cup \cdots \cup a \in WH^{kj}(X)$;

b) for every $h \geq 1$ an Euler class $e(P^h) \in WH^1(X)$ such that $e(P^h)^j$ generates $WH^j(P^h)$;

c) for each $[V] \in WH^k(X), [V'] \in WH^{k'}(Y)$ the cross product $[V] \times [V']$ which is represented by the cartesian product $V \times V'$ (see prop.2);

d) the Künneth isomorphism $WH^n(X \times Y) \cong \sum_{i+j=n} WH^i(X) \bigotimes WH^j(Y)$ which holds because $G = Z_2$ is a field, and furthermore it is defined by the cross product $V \times V' \mapsto V \bigotimes V'$.

The map $\Delta_h : X \times P^h \to X^2 \times S^h/Z_2$ is controlled and hence it induces the homomorphism $\Delta_h^* : WH^{2k}(X^2 \times S^h/Z_2) \to WH^{2k}(X \times P^h)$. Then, given a k-cocycle V we have

$\Delta_h^*(V^2 \times S^h/Z_2) \in WH^{2k}(X \times P^h) \cong \sum_{i+j=2k} WH^i(X) \bigotimes WH^j(P^h)$

and so we can write

$$\Delta_h^*(V^2 \times S^h/Z_2) = \sum_{i+j=2k} R_h^i(V) \times b_j^h e(P^h)^j$$

and define

$$Sq_h^{\alpha}(V) = b_{k-\alpha}^h R_h^{k+\alpha}(V).$$

Notice that

$$\begin{array}{lll} \text{if} \quad h = k-\alpha & \text{then} & \Delta_h^*(V^2 \times S^h/Z_2) = Sq_h^k(V) \times e(P^h)^0 + \cdots + Sq_h^{\alpha}(V) \times e(P^h)^{k-\alpha}\,; \\ \text{if} \quad h = k & \text{then} & \Delta_h^*(V^2 \times S^h/Z_2) = Sq_h^k(V) \times e(P^h)^0 + \cdots + Sq_h^0(V) \times e(P^h)^k\,; \\ \text{if} \quad h > k & \text{then} & \Delta_h^*(V^2 \times S^h/Z_2) = Sq_h^k(V) \times e(P^h)^0 + \cdots + Sq_h^0(V) \times e(P^h)^k + \\ & & \quad + b_{k+1}^h R^{k-1}(V) \times e(P^h)^{k+1} + \cdots + b_{2k}^h R^0(V) \times e(P^h)^{2k} \end{array}$$

but we mean to define independently from $h \in N$

$$Sq^{\alpha}(V) = Sq_h^{\alpha}(V).$$

Proposition. – *The definition of Sq^{α} does not depend on h.*

Proof. – Let $h' \geq h$, and $i : S^h \hookrightarrow S^{h'}$, $I : P^h \hookrightarrow P^{h'}$ be the natural inclusions. Commutativity of diagram (1) implies that of (2)

$$\begin{array}{ccc} X \times P^h & \overset{\Delta_h}{\to} & X^2 \times S^h/Z_2 \\ (1\times I)\downarrow & (1) & \downarrow 1 \times i/Z_2 \\ X \times P^{h'} & \overset{\Delta_{h'}}{\to} & X^2 \times S^{h'}/Z_2 \end{array} \qquad \begin{array}{ccc} WH^{2k}(X^2 \times S^h/Z_2) & \overset{\Delta_h^*}{\to} & WH^{2k}(X \times P^h) \\ (1 \times i/Z_2)^* \uparrow & (2) & \uparrow (1 \times I)^* \\ WH^{2k}(X^2 \times S^{h'}/Z_2) & \overset{\Delta_{h'}^*}{\to} & WH^{2k}(X \times P^{h'}) \end{array}$$

Moreover if $V \in WH^k(X)$ by transversality we find

$$V^2 \times S^h/Z_2 = (1 \times i/Z_2)^{-1}(V^2 \times S^{h'}/Z_2) = (1 \times i/Z_2)^*(V^2 \times S^{h'}/Z_2)$$

and thus

$$\Delta_h^*(V^2 \times S^h/Z_2) = \Delta_h^*(1 \times i/Z_2)^*(V^2 \times S^{h'}/Z_2) = (1 \times I)^*\Delta_{h'}^*(V^2 \times S^{h'}/Z_2) =$$

$$= (1 \times I)^*\Big(\sum_{i+j=2k} R_{h'}^i(V) \times b_j^{h'} e(P^{h'})^j\Big) = \sum_{i+j=2k} b_j^{h'} R_{h'}^i(V) \times I^* e(P^{h'})^j.$$

Now recalling that $I^*(e(P^{h'})^j) = e(P^h)^j$ we can reach the conclusion by the uniqueness of decomposition in

$$WH^{2k}(X \times P^h) \cong \sum_{i+j=2k} WH^i(X) \bigotimes WH^j(P^h)\,.$$

The next proposition is fundamental for showing the Cartan formula but it is convenient to postpone its proof to § 5.

PROPOSITION. – *For every $h > k$ $R_h^{k-1}(V) = \cdots = R_h^0(V) = 0$ and thus independently from $h \geq k$ we have*

$$\Delta_h^*(V^2 \times S^h/Z_2) = \sum_{0 \leq \alpha \leq k} Sq_h^\alpha(V) \times e(P^h)^{k-\alpha}\,.$$

5. The axioms

5.1 – Functoriality

If $f : X \to Y$ is a controlled map between the Whitney spaces X and Y, then $Sq^\alpha \circ f^* = f^* \circ Sq^\alpha$.

Proof. – By hypothesis the map

$$g : X^2 \times S^h/Z_2 \to Y^2 \times S^h/Z_2, \qquad g[x,x,t] = [f(x), f(x), t]$$

is controlled as well. Now, if $V \in WH^k(Y)$ by the transversality lemma we can suppose f transverse to V, and so we find that also g is transverse to $V^2 \times S^h/Z_2$. Therefore we have

$$\begin{aligned} g^*(V^2 \times S^h/Z_2) = [g^{-1}(V^2 \times S^h/Z_2)] = [f^{-1}(V)^2 \times S^h/Z_2] = \\ = \varphi_X[f^{-1}(V)] = \varphi_X f^*(V) \end{aligned}$$

i.e. the commutativity of diagram (1) below.

On the other hand, by the relation of controlled maps $g \circ \Delta_X = \Delta_Y \circ (f \times 1)$ we have the commutativity of diagram (2), and so we conclude putting together diagrams (1), (2), (3) since by definition $Sq^\alpha = pr\, \Delta^* \varphi$.

$$\begin{array}{ccccccc} WH^k(Y) & \overset{\varphi_Y}{\to} & WH^{2k}(Y^2 \times S^h/Z_2) & \overset{\Delta_Y^*}{\to} & WH^{2k}(Y \times P^h) & \overset{pr_Y}{\to} & WH^{k+\alpha}(Y) \\ f^* \downarrow & (1) & g^* \downarrow & (2) & (f \times 1)^* \downarrow & (3) & \downarrow f^* \\ WH^k(X) & \overset{\varphi_X}{\to} & WH^{2k}(X^2 \times S^h/Z_2) & \overset{\Delta_X^*}{\to} & WH^{2k}(X \times P^h) & \overset{pr_X}{\to} & WH^{k+\alpha}(X)\,. \end{array}$$

5.2 – Homomorphism

Each map Sq^α is defined by the composition $Sq^\alpha = pr\Delta^*\varphi$ where pr is the Gysin homomorphism

$$pr_{k+\alpha} : WH^{2k}(X \times P^h) \cong \sum_{i+j=2k} WH^i(X) \bigotimes WH^j(P^h) \to WH^{k+\alpha}(X).$$

Therefore in order to prove that Sq^α is a homomorphism it is sufficient to show that:

Proposition. – *The map $\Delta^*\varphi : WH^k(X) \to WH^{2k}(X^2 \times S^h/Z_2) \to WH^{2k}(X \times P^h)$ is a group homomorphism.*

Proof. – Let V, V' two k-cocycles of X, which we can suppose transverse in X, and observe that with the operation of transverse union in $WH^k(X)$, $\Delta^*\varphi$ is a homomorphism if and only if

$$\Delta^*\Big(\big(V \cup_t V'\big)^2 \times S^h/Z_2\Big) = \Delta^*(V^2 \times S^h/Z_2) + \Delta^*(V'^2 \times S^h/Z_2)\,.$$

Since V and V' are transverse in X, then such are V^2 and V'^2 in X^2 and consequently $V^2 \times S^h/Z_2$, and $V'^2 \times S^h/Z_2$ in $X^2 \times S^h/Z_2$, thus the previous relation becomes

$$\Delta^*((V \cup_t V')^2 \times S^h/Z_2) = \Delta^*(V^2 \times S^h/Z_2 \cup_t V'^2 \times S^h/Z_2)\,.$$

Now we want to subdivide the stratification of $(V \cup_t V')^2 \times S^h/Z_2$.
Recalling that the strata (and so also the costrata) of $V \cup_t V'$ are of the kind $A' - V$, $A - V'$, with $A \subseteq V$ and $A' \subseteq B'$, we obtain the relation

$$(V \cup_t V')^2 = (V^2 \cup_t V'^2) \cup (U \times U' \cup U' \times U)$$

where $U := \overline{V - V'}$ and $U' := \overline{V' - V}$.
Considering the two terms to be the coorientations of (not reduced) cocycles supported in the space $(V \cup V')^2$, (see 2.cor.) we find

$$(V \cup V')^2 = V^2 \cup V'^2 + (U \times U' + U' \times U)$$

These considerations still hold after multiplication by S^h and factorization modulo Z_2-action; therefore

$$(V \cup V')^2 \times S^h/Z_2 = (V^2 \cup_t V'^2) \times S^h/Z_2 + (U \times U' \cup U' \times U) \times S^h/Z_2.$$

Finally using the homomorphism Δ^* and since $\Delta^*((U \times U' \cup U' \times U) \times S^h/Z_2) = 0$ (see the following lemma) we conclude that

$$\begin{aligned}\Delta^*((V \cup_t V')^2 \times S^h/Z_2) &= \Delta^*((V^2 \cup V'^2) \times S^h/Z_2) = \\ &= \Delta^*(V^2 \times S^h/Z_2 \cup_t V'^2 \times S^h/Z_2) = \\ &= \Delta^*(V^2 \times S^h/Z_2) + \Delta^*(V'^2 \times S^h/Z_2).\end{aligned}$$

Lemma. – $\Delta^*((U \times U' \cup U' \times U) \times S^h/Z_2) = 0.$

Proof. – The map $\Delta : X \times P^h \to X^2 \times S^h/Z_2$ is not transverse to $(U \times U' \cup U' \times U) \times S^h/Z_2$. Thus we decompose it into the maps in the following diagram

$$\begin{array}{ccc} X \times P^h & \overset{\Delta'}{\to} & [X^2 \times S^h/Z_2]' \\ & \Delta \searrow & \downarrow I = id \\ & & X^2 \times S^h/Z_2 \end{array}$$

where we have denoted by $[X^2 \times S^h/Z_2]'$ the subdivision of $X^2 \times S^h/Z_2$ induced by the subdivision $[X^2]' = \{S \times T,\ S^2 - d(S),\ d(S)/S, T \text{ strata of } X\}$. Then we find $\Delta^* = \Delta'^* \circ I^*$. Here I^* is a subdivision isomorphism (see 4.2.) whose inverse J acts interpreting the $2k$-cocycles of $[X^2 \times S^h/Z_2]'$ as $2k$-cocycles of $X^2 \times S^h/Z_2$, therefore

$$I^*((U\times U'\cup U'\times U)\times S^h/Z_2) = (U\times U'\cup U'\times U)\times S^h/Z_2 \qquad \text{in} \quad X^2\times S^h/Z_2.$$

Finally since all the restrictions of the map Δ' are diffeomorphisms, and hence they verify all transversality conditions, and this is also true for the diagonal map $d : X \to [X^2]'$, we conclude that

$$\begin{aligned} &\Delta'^* I^*((U \times U' \cup U' \times U) \times S^h/Z_2) = \Delta'^*((U \times U' \cup U' \times U) \times S^h/Z_2) = \\ &= [\Delta'^{-1}((U \times U' \cup U' \times U) \times S^h/Z_2)] = [d^{-1}(U \times U' \cup U' \times U) \times P^h] = 0. \end{aligned}$$

5.3 – Cup product

If V is a k-cocycle of X , $Sq^k(V)$ is the cup product $[V] \cup [V]$.

Proof. – In this case with $\alpha = k$ we can choose $h = k - \alpha = 0$, so that $S^0 \cong P^0 = P$ is a one-point space. Therefore

$$\Delta_0^*(V^2 \times S^h/Z_2) = \Delta_0^*(V^2 \times S^0) = \Delta_0^*(V \times V \times P) = d^*(V \times V) \times id^*[P]$$

and so

$$Sq^k([V]) = d^*([V] \times [V]) = [V] \cup [V]$$

where the last equality holds through singular cohomology.

5.4 – $Sq^0 = identity$

PROPOSITION 1. – *If X is a Whitney space, then the homomorphism Sq^0 is the identity map of $WH^k(X)$.*

Proof. – *step 1*. GORESKY proved in [1] (app.3) that every Whitney space X is deformable into another space having only conical singularities. In particular the deformation provides a Whitney space Y which has a strong deformation retraction onto its subspaces X and X'. The inclusion maps i_X and $i_{X'}$ then induce the isomorphisms i_X^* and $i_{X'}^*$ in singular cohomology and then also in Whitney cohomology. Moreover by functoriality we have the commutative diagram

$$\begin{array}{ccccc} WH^k(X) & \overset{i_X^*}{\leftarrow} & WH^k(Y) & \overset{i_{X'}^*}{\rightarrow} & WH^k(X') \\ Sq_X^0 \downarrow & & Sq_Y^0 \downarrow & & \downarrow Sq_{X'}^0 \\ WH^k(X) & \overset{i_X^*}{\leftarrow} & WH^k(Y) & \overset{i_{X'}^*}{\rightarrow} & WH^k(X') \end{array}$$

and thus it will be sufficient to show that $Sq_{X'}^0 = id$.

step 2. Now if X is a Whitney space having only conical singularities, Goresky proved, again in [1], that X has a subdivision in cells X' which is a Whitney object too. Since the inclusion (identity) map $I : X' \to X$ induces the isomorphism I^* by functoriality $Sq_{X'}^0 I^* = I^* Sq_X^0$ and then it is sufficient to show that $Sq_{X'}^0 = id$.

step 3. Let then X be a Whitney space in which the strata are cells and proceed by induction over $n = \dim X$.

If $\dim X = 0$, then the components of X are points and so we can suppose $X = P$ is a singleton, and $Sq^0 : WH^0(P) \to WH^0(P)$. Here the only non-zero cocycle is the point $V = P$ which obviously verifies the equality

$$\Delta^*((P,P) \times S^h/Z_2) = [\Delta^{-1}(P,P) \times P^h] = [P \times P^h].$$

If $\dim X = n$, then the inclusion of the $(n-1)$-skeleton $i : X_{n-1} \to X_n = X$ induces the homomorphism i^* which is a monomorphism for each $k \leq n$: this follows by the exactness of the sequence of the pair (X, X_{n-1}) in singular theory, and because X being cellular verifies $H^k(X, X_{n-1}) \cong H^k(X/X_{n-1}) \cong 0$.

We then have the commutative diagram

$$\begin{array}{ccc} WH^k(X) & \overset{i^*}{\rightarrow} & WH^k(X_{n-1}) \\ Sq_X^0 \downarrow & & \downarrow Sq_{X_{n-1}}^0 \\ WH^k(X) & \overset{i^*}{\rightarrow} & WH^k(X_{n-1}) \end{array}$$

with i^* monomorphism and $Sq_{X_{n-1}}^0 = id$ by the inductive hypothesis, $Sq_X^0 = 1_{WH^k(X)}$ then follows whenever $k \leq n-1$. It remains to be shown that

$Sq^0_X = 1$ also when $k = n = \dim X$. In such a case every n-cocycle of X is a finite set of points each one of which lies in some stratum of the highest dimension n of X. Writing $V = \{P_1, P_2, \dots P_r\}$ by $[V] = \sum_{i\le r}[P_i]$ and so $Sq^0(V) = \sum_{i\le r} Sq^0(P_i)$ we see that it is sufficient to prove that $Sq^0(P) = P$, as we find by the following proposition.

PROPOSITION 2. – *If P is a point in any n-cell C^n of X then*

$\forall h \ge n \quad \Delta^*((P,P) \times S^h/Z_2) = \{P\} \times e(P^h)^n$ *and in particular* $Sq^0(P) = P$.

Proof. – It will be a posteriori clear that it is not a restriction to identify C^n with R^n.

Let $\Delta^- = \{(x, -x) \mid x \in R^n\} \subseteq R^n \times R^n$ be the antidiagonal space of R^n and let us consider the natural isomorphisms

$$\begin{array}{ccccc} R^n \times R^n \times S^h/Z_2 & \cong & (\Delta^+ \times \Delta^-) \times S^h/Z_2 & \cong & \Delta^+ \times (\Delta^- \times S^h/Z_2) \\ (P,P) \times S^h/Z_2 & \cong & (P,P) \times (0,0) \times S^h/Z_2 & \cong & (P,P) \times ((0,0) \times S^h/Z_2) \end{array}$$

which define a diffeomorphism $G : R^{2n} \times S^h/Z_2 \to \Delta^+ \times (\Delta^- \times S^h/Z_2)$.

Obviously we have $\quad 0 \times S^h/Z_2 \cong 0 \times P^h$ in $\Delta^- \times S^h/Z_2$.

Since $\quad \Delta^- \times S^h/Z_2 \cong \perp (P^h, P^{h+n})$

where the latter is the normal bundle of P^h in P^{h+n}, we can fix a copy P^h_1 of $0 \times P^h$ which is transverse to it. Thus

$$(P,P) \times P^h_1 \quad \text{is transverse to} \quad \Delta^+ \times (0 \times P^h) \text{ in } \Delta^- \times S^h/Z_2$$

i.e.

$$(P,P) \times P^h_1 \quad \text{is transverse to} \quad G \circ \Delta$$

and by the commutativity of the diagram

$$\begin{tikzcd} R^n \times P^h \arrow[r, "\Delta"] \arrow[dr, "d \times I"'] & R^n \times R^n \times S^h/Z_2 \arrow[d, "G"] \\ & \Delta^+ \times (\Delta^- \times S^h/Z_2) \end{tikzcd}$$

we conclude that

$$\begin{aligned}\Delta^*((P,P)\times S^h/Z_2) &= \Delta^* G^*((P,P)\times(0,0)\times S^h/Z_2) =\\ &= (G\circ\Delta)^*((P,P)\times P_1^h) =\\ &= (G\circ\Delta)^{-1}((P,P)\times P_1^h) = d^{-1}((P,P))\times I^{-1}(P_1^h) =\\ &= \{P\}\times e(P^h)^n.\end{aligned}$$

N.B. We can construct directly a cocycle $U=(P,P)\times S^h/Z_2$ transverse to Δ and verifyng $\Delta^{-1}(V)=P\times e^n$ in the following way. If S_1^n is the unit n-sphere in the space $\Delta^-\bigoplus[e_1,e_1]$ and $F_\pm : S_1^n\to S^n$ are the isometries which identify the unit vectors $[e_i,-e_i]\in S_1^n$ with $E_i\in S^n \quad \forall i=1,\dots,n$ and $\pm[e_1,e_1]$ with E_0, writing $S_+^n=\{t|t_0\geq 0\}$, $S_-^n=\{t|t_0\leq 0\}$ then the space $V=graph(F_{+/S_+^n})\cup graph(F_{-/S_-^n})$ is Z_2-equivariant and V/Z_2 is the required cocycle.

Proposition 3. – *For each $h>k$ $\quad R_h^{k-1}(V)=\cdots=R_h^0(V)=0$. Thus independently from h*

$$\Delta_h^*(V^2\times S^h/Z_2)=\sum_{\alpha=0}^{k} Sq^\alpha(V)\times e^{k-\alpha}$$

Proof. – As in steps 1 and 2 of prop.1 we can examine only the case in which X is a cellular Whitney space. But with hypotheses as in step 3 we find that the inclusion of the k-skeleton $i: X_k\to X$ induces $\forall r\leq k$ a monomorphism which makes the following diagram commutative

$$\begin{array}{ccc} WH^k(X) & \xrightarrow{i^*} & WH^k(X_k)\\ R_X^r\downarrow & & \downarrow R_{X_k}^r\\ WH^r(X) & \xrightarrow{i^*} & WH^r(X_k)\,. \end{array}$$

Thus we only have to show that the map $R_{X_k}^r: WH^k(X_k)\to WH^r(X_k)$ is the zero homomorphism $\forall r<k$.

Now, a k-cocycle V of X is a finite set of points which lie in the cells of the highest dimension n of X, and so we can suppose $V=\{P\}$ to be a singleton. Then, since by the previous proposition, $\Delta_h^*((P,P)\times S^h/Z_2)=\{P\}\times e^k$ we can conclude that the only non-zero $R_{X_k}^r$ is for $r=k$.

5.5 – The Cartan formula

Let f, g, Δ, d the following diagonal maps:

$$\begin{cases} f : X^2 \times P^h \to X \times P^h \times X \times P^h, & f(x, x', [t]) = (x, [t], x', [t]); \\ g : X^2 \times X^2 \times S^h/Z_2 \to X^2 \times S^h/Z_2 \times X^2 \times S^h/Z_2, & g([x, x', y, y', t]) = ([x, y, t], [x', y', t]); \\ \Delta = \Delta_{X^2} : X^2 \times P^h \to X^2 \times X^2 \times S^h/Z_2, & \Delta(x, x', [t]) = [(x, x'), (x, x'), t]; \\ d : P^h \to P^h \times P^h, & d([t]) = ([t], [t]). \end{cases}$$

They are clearly all controlled and verify $(\Delta_h \times \Delta_h) f = g\Delta$; thus also $f^*(\Delta_h \times \Delta_h)^* = \Delta^* g^*$.

Lemma 1. – *If V and V' are two cocycles of X then,*

$$g^*(V^2 \times S^h/Z_2 \times V'^2 \times S^h/Z_2) = (V \times V')^2 \times S^h/Z_2.$$

Proof. – This follows directly by preimage once we prove that g is transverse to the given cocycle. But this is true if and only if the map

$g : X^2 \times X^2 \times S^h \to X^2 \times S^h \times X^2 \times S^h$ is transverse to $V^2 \times S^h \times V'^2 \times S^h$

which is easily verified.

Lemma 2. – *For the Euler class $e = e(\gamma_1^{h+1})$, $d^*(e^r \times e^s) = e^{r+s}$ holds.*

Proof. – In singular cohomology by known properties of the vector bundles we have

$$e^r \times e^s = e(\bigoplus_r \gamma_1) \times e(\bigoplus_s \gamma_1) = e\left((\bigoplus_r \gamma_1) \times (\bigoplus_s \gamma_1)\right)$$

So we conclude applying to both members the map d^* which commutes with e.

Proposition. – *For each $V \in WH^n(X)$, $V' \in WH^m(X)$ we have $\forall i = 0, \ldots, n+m$*

$$Sq^i(V \times V') = \sum_{\alpha+\beta=i} Sq^\alpha(V) \times Sq^\beta(V').$$

Proof. – Fix an $h \geq n+m$; we have simultaneously

$$\Delta_h^*((V\times V')^2 \times S^h/Z_2) = \sum_{0\leq i\leq n+m} Sq^i(V\times V')\times e(P^h)^{n+m-i}$$

$$\Delta_h^*(V^2 \times S^h/Z_2) = \sum_{0\leq \alpha\leq n} Sq^\alpha(V)\times e(P^h)^{n-\alpha}$$

$$\Delta_h^*({V'}^2 \times S^h/Z_2) = \sum_{0\leq \beta\leq m} Sq^\beta(V')\times e(P^h)^{m-\beta}.$$

Now by $f = Inv\circ(1_{X^2}\times d)$ through lemma 2 we find

$$f^*(\Delta_h\times\Delta_h)^*(V^2\times S^h/Z_2\times {V'}^2\times S^h/Z_2) =$$
$$= \sum_{\substack{0\leq\alpha\leq n\\ 0\leq\beta\leq m}} Sq^\alpha(V)\times Sq^\beta(V')\times e(P^h)^{n+m-\alpha-\beta}$$

where the first term coincides by lemma 1 with

$$\Delta^* g^*(V^2\times S^h/Z_2\times {V'}^2\times S^h/Z_2) = \Delta^*((V\times V')^2\times S^h/Z_2).$$

Therefore

$$\Delta^*((V\times V')^2\times S^h/Z_2) = \sum_{0\leq i\leq n+m}\left(\sum_{\alpha+\beta=i} Sq^\alpha(V)\times Sq^\beta(V')\times e(P^h)^{n+m-i}\right)$$

and by the uniqueness of decomposition we find

$$Sq^i(V\times V') = \sum_{\alpha+\beta=i} Sq^\alpha(V)\times Sq^\beta(V').$$

REFERENCES

[1] M. Goresky: *Whitney stratified chains and cochains*, Trans.Amer.Math.Soc. **267** (1981), 175-196.

[2] J. Mather: *Notes on topological stability*, Mimeographed notes, Harvard University, 1970.

[3] M. Goresky: *Triangulation of stratified objects*, Proc.Amer.Math.Soc., **72** (1978), 193-200.

[4] T. Brocker and T. Dieck: *Kobordissmen Theorie*, Springer Verlag Lecture Notes, **178**.

[5] S. Buoncristiano, C. Rourke and B. Sanderson: *A geometric approoac to homology theory*, London Math.Soc. Lecture Notes, vol 18, Cambridge Univ.press, New York, 1976.

[6] N. Steenrod and D. Epstein: *Cohomology operations*, Ann.of Math.Studies 50 Princeton Univ. Press. 1962.

[7] D. Quillen: *Elementary Proofs of Some Results of Cobordism Theory Using Steenrod Operations*, Advances in Math., **7** (1971), 29-56.

[8] T. Dieck: *Steenrod-Operationen in Kobordismen-Theorien*, Math.Z, **107** (1968), 380-401.

[9] M. Greenberg and J.Harper: *Algebraic Topology.*

[10] E. Spanier: *Algebraic Topology.*

Pervenuto alla Redazione il 3 marzo 1992