A SURVEY ON STRATIFIED TRANSVERSALITY*

CLAUDIO MUROLO

Centre de Mathématiques et Informatique, Université de Provence,
Laboratoire d'Analyse, Topologie et Probabilités UMR 6632
39, rue Joliot-Curie — 13453 — Marseille — France
murolo@cmi.univ-mrs.fr — http://www.cmi.univ-mrs.fr/~murolo/

In this paper we give a survey on transversality theorems for stratified spaces which have appeared in the literature in the last 30 years having interest for their geometric applications to geometric homology theories.

Keywords: Stratified sets and maps, Stratified transversality, Homology

1. Introduction

We recall that a *stratification* of a topological space A is a locally finite partition Σ of A into C^1 connected manifolds (called the *strata* of Σ) satisfying the *frontier condition* : if X and Y are disjoint strata such that X intersects the closure of Y, then X is contained in the closure of Y. We write then $X < Y$ and $\partial Y = \overline{Y} - Y$ so that $\overline{Y} = Y \sqcup \left(\sqcup_{X<Y} X \right)$ and $\partial Y = \sqcup_{X<Y} X$ ($\sqcup$ = disjoint union).

The pair $\mathcal{X} = (A, \Sigma)$ is called a *stratified space* (or *stratified object)* with *support* A and *stratification* Σ. The union of the strata of dimension $\leq k$ is called the *k-skeleton*, denoted by A_k, inducing a stratified space $\mathcal{X}_k = (A_k, \Sigma_{|A_k})$. A substratified space (or *substratified object* also denoted S.S.O.) of $\mathcal{X}$ is a stratified space $\mathcal{W} = (W, \Sigma_{\mathcal{W}})$, where W is a subset of A, such that each stratum in $\Sigma_{\mathcal{W}}$ is contained in a single stratum of $\mathcal{X}$.

A *stratified map* $f : \mathcal{X} \to \mathcal{X}\,'$ between stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}\,' = (B, \Sigma')$ is a continuous map $f : A \to B$ which sends each stratum X of $\mathcal{X}$ into a unique stratum X' of $\mathcal{X}\,'$, such that the restriction $f_X : X \to X'$ is smooth. We call such a map f a *stratified homeomorphism* if f is a global homeomorphism and each f_X is a diffeomorphism.

*A Zia Carmela

A stratified vector field on $\mathcal{X}$ is a family $\zeta = \{\zeta_X\}_{X\in\Sigma}$ of vector fields, such that ζ_X is a smooth vector field on the stratum X.

Extra conditions may be imposed on the stratification Σ, such as to be an *abstract stratified set* in the sense of Thom–Mather [12, 21, 22] or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney [21, 22, 42], or (c) of K. Bekka [3] or, when A is a subset of a C^2 manifold, to satisfy conditions (w) of Kuo–Verdier [41], or (L) of Mostowski [38].

We send the reader to the original papers and to the above references for their definitions and main properties.

Let $\mathcal{X} = (A, \Sigma)$ be a stratified space. For stratified transversality we mean the problem of deforming a substratified object $\mathcal{W}$ of $\mathcal{X}$ via a stratified isotopy $\Phi : \mathcal{X} \times I \to \mathcal{X}$ to a substratified object $\mathcal{W}' = \Phi_1(\mathcal{W})$ of $\mathcal{X}$ which is transverse to $\mathcal{V}$ in $\mathcal{X}$, or more generally with respect to a fixed stratified map $g : \mathcal{Y} \to \mathcal{X}$, for some stratified space $\mathcal{Y}$.

This problem was solved by Clint McCrory for stratified polyhedra [23, 24] ; his result is essential to the foundations of intersection homology [16].

For Whitney (b)-regular stratifications Mark Goresky gave a transversality theorem ([15], 5.3) valid only for π-*fibre* substratified objects $\mathcal{W}$, and *controlled maps* $g : \mathcal{Y} \to \mathcal{X}$ [21, 22]. These two properties mean respectively that the support W of $\mathcal{W}$ is locally, near each point x of A, a union of fibres of a projection $\pi_S : T_S \to S$ where S is the stratum of $\mathcal{X}$ containing x and a similar property at the level of the fibre of g. This result is essential in proving the main theorems of Goresky about representing the cohomology by stratified objects ([15] 4.7 and 6.2).

More recently in [31] and [32], A. du Plessis, D. Trotman and myself gave two different proofs that *"after stratified isotopy of $\mathcal{X}$, a stratified subspace $\mathcal{W}$ of $\mathcal{X}$, or a stratified map $h : \mathcal{Z} \to \mathcal{X}$, can be made transverse to a fixed stratified map $g : \mathcal{Y} \to \mathcal{X}$"* : the second (historically the first, [29] 1997) using time-dependent vector field techniques and the family of geodesics introduced by Mather [20] and the first (historically the second) by adapting some ideas indicated at the end of Goresky's thesis ([14], 1976) and in the 1987 book [17]. The authors of [31, 32] obtained a generalisation of Goresky's theorem, with less restrictive hypotheses and which applies to all abstract stratified sets and (w)-regular nice stratifications, hence for any (b)-, (c)- or (L)-regular nice stratification, and which allows one to develop further Goresky's geometric homology theory [29]. This stratified transversality theorem holds for the most important types of regular stratifications, and for every stratified map without assuming control conditions.

In particular, we obtain an analogue of Goresky's theorem for stratified maps $g : \mathcal{Y} \to \mathcal{X}$ which are not necessarily controlled and for substratified objects $\mathcal{W}$ which are not necessarily π-fibre.

The analogous theorem for two stratified maps was also obtained.

We present moreover various applications of Goresky's transversality theorem and of its generalisations in [31, 32], pointing out some related problems which are still open.

I thank David Trotman who suggested I write this survey.

2. The PL-Transversality Theorem

In the context of PL-stratified spaces, the problem of putting in transverse position two substratified polyedra of a stratified polyedron $\mathcal{X}$ was solved in 1977 by Clint McCrory [24]. This theorem states :

Theorem 2.1. *Let $\mathcal{X}$ be a stratified polyedron and A, B, C closed subpolyedra with $B \supseteq C$. There exists a PL isotopy $H : \mathcal{X} \times I \to I$ such that :*
i) $|H_t(x) - x| < \epsilon$ for all x and $t \in I$;
ii) $H_t(x) = x$ for all $x \in C$ and all $t \in I$;
iii) A and $H_1(B - C)$ are in general position in $\mathcal{X}$.

Such a theorem, which uses a simplicial technique of Zeeman, was first proved without the property ii) by McCrory in his Ph.D. thesis ([23] p. 98, 1972) and previously again without the property i) in Akin's Ph.D. thesis ([2] p. 471, 1969).

Recall historically that in 1895 and 1899, in his famous papers *Analysis Situs* and *Complément à l'Analysis Situs* which founded modern algebraic topology, H. Poincaré studied the intersection of an i-cycle and a j-cycle in a compact oriented n-manifold $\mathcal{X}$ in the case of complementary dimension ($i + j = n$) then, in 1926, the theory was extended by S. Lefschetz to arbitray i and j. Fifty years later in 1980, in their celebrated paper [16] Mark Goresky and Robert MacPherson introduced *Intersection Homology Theory* in which they generalize to a class of singular spaces, the PL-pseudomanifolds, the Poincaré–Lefschetz cup product pairing $\cap : H_i(X) \times H_j(X) \to H_{i+j-n}(X)$.

In this context, McCrory's PL-transversality theorem (in a relative version), played a fundamental role in defining the Goresky–MacPherson intersection pairing which extends the Poincaré–Leschetz map and was in this way essential to the foundation of intersection homology theory.

3. Transversality for Whitney stratifications

After the years 1965–70, during which H. Whitney laid the foundations of Whitney stratification theory [42] and R. Thom and J. Mather ([39] and

[21, 22]) those of *abstract stratified sets* as the larger class of desirable singular spaces, M. Goresky was preparing his Ph.D. thesis directed by R. MacPherson. The goal of his thesis, *Geometric Cohomology and Homology of Stratified Objects* [14], was to introduce new geometric homology and cohomology theories, for a stratified space $\mathcal{X}$, in which cycles and cocycles of $\mathcal{X}$ could be *represented* by substratified spaces with the same type of singularity as $\mathcal{X}$. In this theory a well adapted stratified transversality theorem would allow to find the geometric cap product as intersection of a cycle and a cocycle in transverse position and many other very nice geometric interpretations of the algebraic operations (see also §4).

We will talk later on about the stratified transversality results, statements and techniques which one can find in the thesis of Goresky, an exciting source of interesting results and nice and useful ideas.

We will present first the geometric homology theories published in 1981 in *Whitney Stratified Chains and Cochains* [15], in a revised version with respect to the thesis of 1976, and the stratified transversality theorem underlying this revised theory of 1981.

In this 1981 paper, Goresky re-defines his geometric homology and cohomology theories only for Whitney (b)-regular stratifications (and not for abstract stratified sets) and gives a new completely revised version of the previous stratified transversality statements and proofs.

For $\mathcal{X} = (A, \Sigma)$ a (b)-regular stratified space of support $A \subseteq \mathbb{R}^n$, Goresky introduces the *homology and cohomology sets* $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$ (called *Whitney homology and cohomology theory* from [27]): the elements of $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$ are equivalence classes of Whitney substratified k-cycles and k-cocycles of $\mathcal{X}$ with respect to a *Whitney stratified cobordism*.

A *Whitney stratified k-cycle* $\xi = (\mathcal{V}, z)$ of $\mathcal{X}$ is a compact (b)-regular k-substratified object $\mathcal{V}$ of $\mathcal{X}$ together with an *orientation of* $\mathcal{V}$, that is an element $z = \sum_j n_j V_j^k$ of the free abelian group $C_k(\mathcal{V})$ on $\mathbb{Z}$ on the oriented k-strata $\{V_j^k\}_j$ of $\mathcal{V}$ whose *boundary* is 0.

A cobordism between two stratified k-cycles ξ and ξ' is defined as a (b)-regular $(k+1)$-S.S.O. ζ of $\mathcal{X} \times I$ (I stratified by $\{\{0\},]0,1[, \{1\}\}$) with boundary $\partial\zeta = \xi \times \{0\} - \xi' \times \{1\}$ (modulo *reduction*[a]).

A *substratified object* $\mathcal{W}$ of $\mathcal{X}$ satisfies the π-*fibre condition* (with respect

[a]The reduction of a chain or of a cochain $\xi = (V, z)$ with $z = \sum_j n_j V_j^k$ is the chain or the cochain $\xi_{/} = (V_{/z}, z)$ where $V_{/z} = \overline{\cup_{n_j \neq 0} V_j^k} = \cup_{n_j \neq 0} \overline{V_j^k}$. Every chain and cochain is identified with its reduction.

to a fixed system of control data of $\mathcal{X}$, $\mathcal{F} = \{(\pi_S, \rho_S, T_S) | S \text{ stratum of } \mathcal{X}\})$ if there exists an $\epsilon > 0$ such that for every stratum S of $\mathcal{X}$ one has : $\pi_S^{-1}(W) \cap T_S(\epsilon) = W \cap T_S(\epsilon)$. This condition allows Goresky to define for each k-costratum D (i.e. a connected component of the union of all the strata embedded in $\mathcal{X}$ with codimension k) a tubular neighbourood T_D of D in A. Thinking of T_D as a normal fiber bundle of D in $\mathcal{X}$, an orientation of its unit sphere bundle defines a *coorientation* of D.

A *Whitney stratified k-cocycle* $\theta = (\mathcal{W}, c)$ of $\mathcal{X}$ is then defined as a π-fibre Whitney substratified object $\mathcal{W}$ of $\mathcal{X}$, embedded in $\mathcal{X}$ with codimension k together with a *k-coorientation of* $\mathcal{W}$, that is an element $c = \sum_s n_s D_s^k$ of the free abelian group $C^k(\mathcal{W})$ on the oriented k-costrata $\{D_s^k\}_s$ of $\mathcal{W}$, and whose *coboundary* is 0. A *cobordism* between two stratified k-cocycles θ and θ' is defined as a (b)-regular $(k+1)$-π-fibre S.S.O. θ of $\mathcal{X} \times [0, 1]$ with boundary $\delta\theta = \theta \times \{0\} - \theta' \times \{1\}$ (modulo *reduction*(*)).

The fundamental reason for which such homology and cohomology sets exist is that every (b)-regular S.S.O. $\mathcal{V}$ admits a system of control data [21]. This again works by considering for $\mathcal{X}$ and $\mathcal{V}$ abstract S.O. (see [14] and [29, Chapter IV, p. 134] for details).

To simplify the notations, we will omit the orientation z of a cycle ξ and the coorientation c of a cocycle θ and will write $\mathcal{V}$ or V (the support of $\mathcal{V}$) for ξ and $\mathcal{W}$ or W (support of $\mathcal{W}$) for θ.

Goresky introduced the two *homology and cohomology representation maps*

$$R_k : WH_k(\mathcal{X}) \to H_k(A) \quad \text{and} \quad R^k : WH^k(\mathcal{X}) \to H^k(A)$$

analogues of the Thom–Steenrod map between the differential bordism and the singular homology of $\mathcal{X} = (A, \Sigma)$.

The stratified transversality theorem underlying this theory is Goresky's *"Transversality Lemma"* (5.3 [15]) that we include with its original proof and notations:

Theorem 3.1. *Suppose $\mathcal{X}_1$ and $\mathcal{X}_2$ are Whitney stratified subsets of two manifolds M_1 and M_2 (respectively). Fix a system of control data on $\mathcal{X}_1$ and $\mathcal{X}_2$ and let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a stratified map. Suppose $\mathcal{Y} \subseteq \mathcal{X}_2$ is a geometric cocycle. Suppose either (a) f is controlled or (b) f is the restriction of a smoth map $\tilde{f} : M_1 \to M_2$. Then $\mathcal{Y}$ is cobordant to a cocycle $\mathcal{Y}' \subseteq \mathcal{X}_2$ such that f is transverse to $\mathcal{Y}'$.*

Proof. Assume by induction on k that $\mathcal{Y}$ is cobordant to a cocycle $\mathcal{Y}_k$ such that f is transverse to $\mathcal{Y}_k \cap (\mathcal{X}_2)_k$ where $(\mathcal{X}_2)_k$ denotes the k-skeleton of

$\mathcal{X}_2$. We will find a controlled vector field η on $\mathcal{X}_2$ with (controlled) flow $F_t : \mathcal{X}_2 \to \mathcal{X}_2$ such that $\mathcal{Y}_{k+1} = F_1(\mathcal{Y}_k)$ satisfies the induction hypothesis.

Under either assumption (a) or (b) above there is a neigbourhood U of $(\mathcal{X}_2)_k$ such that f is transverse to $\mathcal{Y}_k \cap U$.

Let $S = (\mathcal{X}_2)_k - (\mathcal{X}_2)_{k-1}$, by Sard's Theorem [13] there is a controlled vector field η_S on S with time 1 flow $F_S : S \to S$ such that η_S vanishes near $(\mathcal{X}_2)_k$ and such that f is transverse to $F_S(\mathcal{Y}_k)$. Take η to be any controlled lift [12], [21] of η_S. $\square$

This proof was for a long time and until 2000, not understandable to me (and to my knowledge to various other mathematicians). Thus I talk in the introduction of my Ph.D. thesis [29] 1997, of a *"mistake in the Goresky proof . . ."*. But after my joint work with D. Trotman and A. du Plessis [31] I no longer think there is a mistake.

The main reason for which this proof was obscure to me was the fact that if we would *first* obtain a transversalizing map $F_S : S \to S$ such that $\mathcal{Y}_{k+1} = F_S(\mathcal{Y}_k)$ is transverse to f, then *after* there is no way to replace the choice of F_S in some open dense set of diffeomorphisms to obtain that *(*): "F_S is also the time 1 flow F_1 of a vector field (not depending on time)"*.

This impossibility comes from a theorem of C. Freifeld [10] who remarked first a phenomenon better explained later by J. Milnor [25] 1980 that *"in the infinite dimensional space $Diff(S, S)$ the maps having the property (*) above, i.e. lying in a one parameter group of diffeomorphisms of S, do not fill a neighbourhood of the identity 1_S"* (more about this difficulty in the introduction of [29]).

On the other hand one could say : in the aim of Goresky the map F_S has to be obtained *at the same time as* the vector field η_S of which F_S is the time 1 flow.

But then how to do it ? In this sense the reference given by Goresky [13] on Sard's theorem is really not clear and very far from orienting the reader to an understandable continuation of the proof. Also remark that for a vector field η to have a flow defined $\forall t \in I$ is equivalent to asking that its flow is defined for every $t \in \mathbb{R}$ (i.e. that η is complete).

Such difficulties, motivated me jointly with A. du Plessis and D. Trotman in 2001 and 2005 [31, 32] to find two new and different generalisations of this stratified transversality theorem : the first by considering (for the η_S) time-dependent vector fields (whose flows are not necessarily one parameter groups !) and the second by putting together some methods sketched in the appendix of Goresky's thesis [14] and in his 1987 book [17] which uses a very fine idea of Abraham [1] to apply Sard's theorem on the space of vector fields on a manifold S and about which we will come back later.

After having established the transversality theorem underlying his whole theory Goresky deduces first of all the important theorem which gives the bijectivity of the cohomological representation.

Theorem 3.2. *For every Whitney stratification $\mathcal{X}$, the cohomology representation map $R^k : WH^k(\mathcal{X}) \to H^k(\mathcal{X})$ is a set bijection.*

The corresponding homology theorem below does not follow from the transversality theorem, however the proof (of the relative to the boundary version) was important in proving Theorem 3.2 above so we like to recall it underlining that it was proved by Goresky only for $\mathcal{X}$ the trivial stratification of a manifold.

Theorem 3.3. *If $\mathcal{X} = \{M\}$ is a trivial stratification of a manifold possibly with boundary, the homology representation map $R_k : WH_k(\mathcal{X}) \to H_k(M)$ is a set bijection.*

The same statement for $\mathcal{X}$ an arbitrary (b)-regular stratification :

Conjecture 3.1. *If $\mathcal{X}$ is a Whitney stratification the homology representation map $R_k : WH_k(\mathcal{X}) \to H_k(M)$ is a set bijection.*

remains a famous problem of Goresky (thesis [14] 1976 and [15] 1981) still unsolved. On the other hand, one easily sees that, if one proves the celebrated conjecture :

Conjecture 3.2. *Every Whitney stratification $\mathcal{X}$ admits a Whitney triangulation.*

this will also give a solution of the above conjecture 3.1 of Goresky.

Then, using also the following proposition :

Proposition 3.1. *Suppose $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a controlled stratified map and $\mathcal{Y}$ is a codimension k geometric cocycle in $\mathcal{X}_2$ such that f is transverse to $\mathcal{Y}$. Then $f^{-1}(\mathcal{Y})$ is a π-fibre subset of $\mathcal{X}_1$ which admits a canonical Whitney stratification. The coorientation of $\mathcal{Y}$ pulls back to a coorientation on $f^{-1}(\mathcal{Y})$ which then becomes a geometric cocycle $f^{-1}(\mathcal{Y})$ and it represents the cohomology class $f^*([\mathcal{Y}])$ in $H^k(\mathcal{X}_1)$.*

Goresky's transversality theorem, again makes richer this geometric theory. It also allows Goresky to prove first the following theorem on the cohomology cup product :

Theorem 3.4. *Suppose $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are geometric cocycles in a Whitney object $\mathcal{X}$. Then $\mathcal{Y}_2$ is cobordant to a cocycle $\mathcal{Y}'_2$ which is transverse to $\mathcal{Y}_1$. In this case $\mathcal{Y}_1 \cap \mathcal{Y}'_2$ is a geometric cocycle with the product coorientation and $[\mathcal{Y}_1 \cap \mathcal{Y}'_2] = [\mathcal{Y}_1] \cup [\mathcal{Y}'_2]$.*

and then the proposition stating that also the cap product has, thanks to the transversality theorem, a nice geometric meaning :

Proposition 3.2. *Suppose $\mathcal{Y}$ is a geometric k-cocycle in $\mathcal{X}$ and $\mathcal{Z}$ is a geometric p-cycle. Then $\mathcal{Y}$ is cobordant to a cocycle $\mathcal{Y}'$ which is transverse to $\mathcal{Z}$. Using the product orientation on $(p-k)$-strata of $\mathcal{Y}' \cap Z$ (which all have the form (p-costratum of $\mathcal{Y}$) $\cap$ (k-stratum of $\mathcal{Z}$)), $\mathcal{Y}' \cap \mathcal{Z}$ becomes a geometric cycle and it represents the cap product $[\mathcal{Y}] \cap [\mathcal{Z}] \in H_{p-k}(\mathcal{X})$.*

4. Further geometric applications of the Goresky Theorem

Using again the transversality theorem of Goresky, in 1994 [27], I improved the Goresky theories by introducing in the homology and cohomology *sets* $WH_*(\mathcal{X})$ and $WH^*(\mathcal{X})$ a geometric sum operation which geometrically means *transverse union* of cycle and/or of cocycles below denoted by $\cup_t$ (see §5.2 for the rigorous definitions of $\cup_t$ and $\cap_t$). This was done in the same spirit as for the *Moving Lemma* in the Chow Group theory for the algebraic cycles of an algebraic manifold [11] and in such a way that the sets $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$ become abelian groups and the representation maps R_k and R^k group isomorphisms.

Again in the homology case the full theorem was obtained only when $\mathcal{X} = \{M\}$ reduces to a smooth manifold.

In [27] I complete the Goresky theories with a slight algebraization and by introducing the coefficients in an abelian group G and in [27, 28] showing that the most important cohomology operations, *the Steenrod squares* and *the Steenrod p-powers* (p an odd prime) can be realized through a geometric construction based on transversality methods. Then starting from [27, 28] I refer to WH_* and WH^* as *Whitney Homology and Cohomology theories.*

All geometric applications of the Goresky transversality theorem (rewriting more nicely his theorems on the cup and cap product), together with the improvements in [27, 28] can be summarized as follows :

Theorem 4.1. *If $\mathcal{X}$ is a smooth manifold, the sum operation defined in $WH_k(\mathcal{X})$ by :*

$$+ \quad : \quad WH_k(\mathcal{X}) \times WH_k(\mathcal{X}) \to WH_k(\mathcal{X}) \quad , \quad [\mathcal{V}_1] + [\mathcal{V}_2] = [\mathcal{V}_1 \cup_t \mathcal{V}'_2]$$

where $\mathcal{V}'_2$ is a cycle cobordant to $\mathcal{V}_2$ and transverse to $\mathcal{V}_1$ (which exists by the transversality theorem) is a well defined group operation in $WH_k(\mathcal{X})$ for which the Goresky homology representation map $R_k : WH_k(\mathcal{X}) \to H_k(\mathcal{X})$ is a group isomorphism.

Theorem 4.2. *For every Whitney object $\mathcal{X}$ the sum operation defined in $WH^k(\mathcal{X})$ by:*

$$+ \quad : \quad WH^k(\mathcal{X}) \times WH^k(\mathcal{X}) \to WH^k(\mathcal{X}) \quad , \quad [\mathcal{Y}_1] + [\mathcal{Y}_2] = [\mathcal{Y}_1 \cup_t \mathcal{Y}'_2]$$

where $\mathcal{Y}'_2$ is a coycle cobordant to $\mathcal{Y}_2$ and transverse to $\mathcal{Y}_1$ (which exists by the transversality theorem) is a well defined group operation in $WH^k(\mathcal{X})$ and the Goresky cohomology representation map $R^k : WH^k(\mathcal{X}) \to H^k(\mathcal{X})$ is a group isomorphism.

Theorem 4.3. *For every stratified controlled map $f : \mathcal{X}_1 \to \mathcal{X}_2$ between Whitney stratifications there is an induced map f^* in Whitney cohomology defined by :*

$$f^* : WH^k(\mathcal{X}_2) \to WH^k(\mathcal{X}_1) \quad , \quad f^*([\mathcal{Y}]) = [f^{-1}(\mathcal{Y}')]$$

where $\mathcal{Y}'$ is cobordant to $\mathcal{Y}$ and transverse to f, and exists by the transversality theorem (f^* is given by the transverse preimage of a geometric cocycle). Moreover with respect to the transverse sum in $WH^k(\mathcal{X}_2)$ and $WH^k(\mathcal{X}_1)$, $f^* : WH^k(\mathcal{X}_2) \to WH^k(\mathcal{X}_1)$ is a group homomorphism.

Theorem 4.4. *In the geometric cohomology theory WH^* the cup product is defined by*

$$\cup \quad : \quad WH^*(\mathcal{X}) \times WH^*(\mathcal{X}) \to WH^*(\mathcal{X}) \quad , \quad [\mathcal{Y}_1] \cup [\mathcal{Y}_2] = [\mathcal{Y}_1 \cap_t \mathcal{Y}'_2]$$

where $\mathcal{Y}'_2$ is cobordant to $\mathcal{Y}_2$ and transverse to $\mathcal{Y}_1$ and exists by the transversality theorem. I.e. the cup product is given by the transverse intersection of two geometric cocycles.

Proposition 4.1. *In the geometric theory WH_*, WH^* the cap product is defined by*

$$\cap \quad : \quad WH_{p+k}(\mathcal{X}) \times WH^k(\mathcal{X}) \to WH_p(\mathcal{X}) \quad , \quad [\mathcal{V}] \cap [\mathcal{Y}] = [\mathcal{V} \cap_t \mathcal{Y}']$$

where $\mathcal{Y}'$ is cobordant to $\mathcal{Y}$ and transverse to $\mathcal{V}$ and exists by the transversality theorem. I.e. the cap product is the transverse intersection of a cocycle with a cycle.

Finally, although such a nice geometric interpretation of the cross product [15] and of Poincaré Duality [29] does not come from transversality we like to recall it :

Proposition 4.2. *In cohomology WH^* the cross product of cocycles is defined by*

$$\times \quad : \quad WH^k(\mathcal{X}) \times WH^h(\mathcal{Y}) \to WH^{k+h}(\mathcal{X}\times\mathcal{Y}) \quad , \quad [\mathcal{Y}_1]\times[\mathcal{Y}_2] = [\mathcal{Y}_1\times\mathcal{Y}_2]\,.$$

I.e. the cross product is the cartesian product of two cocycles of $\mathcal{X}$ and $\mathcal{Y}$.

Proposition 4.3. *If $\mathcal{X}$ is a n-manifold the (inverse map of the) Poincaré Duality isomorphism $\mathcal{D}$ is the "identity" on the representative cocycles :*

$$\mathcal{D} : WH^k(\mathcal{X}) \to WH_{n-k}(\mathcal{X}) \quad , \quad \mathcal{D}([\mathcal{V}]) = [\mathcal{V}]\,.$$

I.e. it is obtained by interpreting each k-cocycle as a $(n-k)$-cycle of $\mathcal{X}$.

This nice geometric interpretation exists still when $\mathcal{X}$ is not a manifold but just a pseudomanifold (because in this case the the fundamental class $[\mathcal{X}]$ is a cycle of $\mathcal{X}$).

Finally, in Whitney cohomology there exists a geometric realisation of the main cohomology operations, the Steenrod Squares $\{Sq^\alpha\}_\alpha$ [27] and the Steenrod p-powers $\{P^\alpha\}_\alpha$ (p prime odd) [28] based on stratified transversality methods.

Theorem 4.5. *Let $A : \mathcal{X}^2 \times S^h \to \mathcal{X}^2 \times S^h$ be the $\mathbb{Z}_2$-action defined by $A(x,y,t) = (y,x,-t)$), φ the map $\varphi([W]) = [(W^2 \times S^h)/\mathbb{Z}_2]$, Δ the map $\Delta(x,[t]) = [((x,x),t)]$ and for every α, $pr_{k+\alpha}$ the Gysin homomorphism.*

Then independently from $h \geq k$, the composition map:

$$Sq^\alpha : \quad WH^k(\mathcal{X}) \quad \xrightarrow{\varphi} \quad WH^{2k}((\mathcal{X}^2\times S^h)/\mathbb{Z}_2) \quad \xrightarrow{\Delta^*} \quad WH^{2k}(\mathcal{X}\times\mathbb{P}^h) \cong$$

$$\cong \sum_{i+j=2k} WH^i(\mathcal{X}) \otimes WH^j(\mathbb{P}^h) \xrightarrow{pr_{k+\alpha}} WH^{k+\alpha}(\mathcal{X})$$

is a geometric construction of the Steenrod squares $\{Sq^\alpha : WH^k(\mathcal{X}) \to WH^{k+\alpha}(\mathcal{X})\}_{X,\alpha,k}$ in Whitney cohomology. I.e. we have :

1) Sq^α is a group homomorphism ;
2) $f : \mathcal{X} \to \mathcal{Y}$ is a controlled map $\quad \Rightarrow \quad Sq^\alpha f^* = f^* Sq^\alpha$;
3) $k = \alpha \Rightarrow Sq^\alpha([W]) = [W] \cup [W]$ is the cup product ;
4) $Sq^0 = 1_{WH^k(X)}$ is the identity map ;
5) $Sq^i([W] \times [W']) = \sum_{\alpha+\beta=i} Sq^\alpha([W]) \times Sq^\beta([W'])$;
6) $\alpha > k \Rightarrow Sq^\alpha([V]) = 0$.

Theorem 4.6. *Let $p \in \mathbb{N}$ be an odd prime, $h \in \mathbb{N}$, odd so $S^h \subseteq \mathbb{C}^r$, $r = (h+1)/2$ and $L_p = S^h/\mathbb{Z}_p$ be the Lens space quotient of S^h with respect to the multiplication for $e^{i\frac{2\pi}{p}}$ in $\mathbb{C}^r$.*

Let $A : \mathcal{X}^p \times S^h \rightarrow \mathcal{X}^p \times S^h$ be the $\mathbb{Z}_p$-action defined by $A((x_1, \ldots, x_p), z) = ((x_p, x_1 \ldots, x_{p-1}), e^{i\frac{2\pi}{p}} z)$, φ the map defined by $\varphi([W]) = [(W^p \times S^h)/\mathbb{Z}_p]$, Δ the map $\Delta(x, [z]) = [((x, \ldots, x), z)]$ and for every $\alpha' = k + 2\alpha(p-1)$, $pr_{\alpha'}$ the Gysin homomorphism.

Then, independently from $h \geq (k - 2\alpha)(p-1)$, the composition map:

$$P^\alpha : \quad WH^k(\mathcal{X}) \xrightarrow{\varphi} WH^{kp}((\mathcal{X}^p \times S^h)/\mathbb{Z}_p) \xrightarrow{\Delta^*} WH^{kp}(\mathcal{X} \times L_p^h) \cong$$

$$\cong \sum_{i+j=kp} WH^i(\mathcal{X}) \otimes WH^j(L_p^h) \overset{pr_{\alpha'}}{\rightarrow} WH^{\alpha'}(\mathcal{X})$$

is a geometric construction of the Steenrod p-powers $\{P^\alpha : WH^k(\mathcal{X}) \rightarrow WH^{\alpha'}(\mathcal{X})\}_{X,\alpha,k}$ in Whitney cohomology. I.e. we have :

1) P^α is a group homomorphism ;

2) $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a controlled map $\Rightarrow$ $P^\alpha f^* = f^* P^\alpha$;

3) $k = 2\alpha \Rightarrow P^\alpha([W]) = [W] \cup [W] \cup \cdots \cup [W]$ is the cup product k times of $[W]$;

4) $P^0 = 1_{WH^k(X)}$ is the identity map ;

5) $P^i([W] \times [W']) = \sum_{\alpha+\beta=i} P^\alpha([W]) \times P^\beta([W'])$;

6) $2\alpha > k \Rightarrow P^\alpha([W]) = 0$.

5. Improvement of the Goresky Theorem. Applications. Open problems

Goresky's transversality theorem applies to those substratified objects $\mathcal{W}$ of $\mathcal{X}$ satisfying a π-*fibre condition* with respect to a fixed system of control data $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \rightarrow X \times [0, \infty)\}_{X \in \Sigma}$ of $\mathcal{X}$, and to a stratified map $g : \mathcal{Y} \rightarrow \mathcal{X}$ which is controlled with respect to two systems of control data.

The π-fibre condition says that $\mathcal{W}$ is locally, near each point x of A, a union of fibres of the projection $\pi_S : T_S \rightarrow S$ where S is the stratum containing x, while the control condition on the map g imposes a similar property for the fibres of g (and of π_S). These conditions were used by Goresky to preserve transversality with respect to g of a deformation $\mathcal{W}'$ of $\mathcal{W}$ in his inductive proof. As explained above, Goresky's transversality theorem has been shown to be very useful in several important applications [15], [27, 28]; but the hypotheses of π-fibre on $\mathcal{W}$ and control on g prevent a wider use.

Definition 5.1. Let $\mathcal{X} = (A, \Sigma)$ be a stratified space. A *stratified isotopy* of $\mathcal{X}$ (or of A) $\Phi : A \times I \to A$ (denoted also $\{ \Phi_t : A \to A \}_{t \in I}$) is a stratified map such that for every $t \in I$, the map *at time* t, $\Phi_t : A \to A$ is a stratified homeomorphism.

Clearly, if $\{\Phi_t\}_{t\in I}$ and $\{\Psi_t\}_{t\in I}$ are stratified isotopies, so is $\{\Psi_t \circ \Phi_t\}_t$.

Definition 5.2. Let $\mathcal{W} = (W, \Sigma_{\mathcal{W}})$ and $\mathcal{W}' = (W', \Sigma_{\mathcal{W}'})$ be two S.S.O. of a stratified space $\mathcal{X} = (A, \Sigma)$. We say that $\mathcal{W}'$ *is a deformation by isotopy of* $\mathcal{W}$ *in* A if there exists a stratified isotopy $\Phi : A \times I \to A$ such that $\Phi_0 = 1_A$ and $\mathcal{W}' = \Phi_1(\mathcal{W})$.

If $\Phi : A \times I \to A$ is a stratified isotopy of $\mathcal{X} = (A, \Sigma)$ and $\mathcal{W}$ is a substratified object of $\mathcal{X}$, then for each $t \in I$ the image $\mathcal{W}' = \Phi_t(\mathcal{W})$ is a substratified object with stratification induced by Φ_t and $\mathcal{W}'$ is a deformation by isotopy of $\mathcal{W}$.

Let $h, h' : \mathcal{Y} \to \mathcal{X}$ be two stratified maps. We say that h' *is a deformation by isotopy of* h *in* $\mathcal{X}$ if there exists a stratified isotopy $\Phi : \mathcal{X} \times I \to \mathcal{X}$ such that $\Phi_0 = 1_{\mathcal{X}}$ and $h' = \Phi_1 \circ h$, i.e. h' is the deformation via Φ and at time $t = 1$ of h.

"Deformation by isotopy" of S.S.O. of $\mathcal{X}$ and of maps $h, : \mathcal{Y} \to \mathcal{X}$ define clearly equivalence relations.

In two recent papers [31, 32], A. du Plessis, D. Trotman and myself, gave two different proofs of the stratified transversality theorem below, of which we recall here the ideas of the proofs.

Theorem 5.1. *Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set, or a (w)-regular nice stratified subset of a manifold, and let $g : \mathcal{Y} \to \mathcal{X}$ be a stratified map.*

Then for each stratified map $h : \mathcal{Z} \to \mathcal{X}$ and each open neighbourhood U of $h(\mathcal{Z})$ in $\mathcal{X}$, there exists a deformation by isotopy h' of h in $\mathcal{X}$ which is transverse to g in $\mathcal{X}$ and such that $h'(\mathcal{Z}) \subseteq U$. If C is a closed subset of $\mathcal{X}$ on which h is transverse to g then one can obtain that $h' = h$ on C.

Proof. Both proofs are given by induction on the dimension $k \leq n = \dim \mathcal{X}$ of the skeleton $\mathcal{X}_k$ of $\mathcal{X}$ by constructing a stratified vector field $\zeta = \zeta_k$ of $\mathcal{X}$ having a time 1 flow $\Phi_1 = \Phi_1^k$ defined on the whole of $\mathcal{X}$ and such that the map $h'_k = \Phi_1 \circ h$ satisfies the inductives hypotheses.

To obtain this, ζ has to be 0 on $\mathcal{X}_{k-1}$ (so $\Phi_{1|\mathcal{X}_{k-1}} = id$) and the restriction $f = \Phi_{1\,S} : S \to S$ where $S = \mathcal{X}_k - \mathcal{X}_{k-1}$ has to be a diffeomorphism of S such that $f \circ h$ is transverse to g.

Outlines of proof in [32]. We first prove that the set of such diffeomorphisms f of S is open and dense in the connected component $Diff_0(S,S)$ of 1_S in $Diff(S,S)$. Then we apply the techniques used by Mather [20] to show that infinitesimal stability implies stability, and using the families of geodesics of S we prove that *"There exists a (sufficiently small) neighbourhood U' of $1_S \in Diff_0(S,S)$ such that every $f \in U'$ is the time 1 flow $f = \Phi_1$ of a time-dependent vector field $\zeta = \zeta(x,t)$ such that $\lim_{x \to X_{k-1}} \zeta(x,t) = 0$"* (we stress that this property is completely false [10] without the precision time-dependent as we said in §3).

This allows us to obtain a *"Stratified Extension Theorem"* [32] holding for the diffeomorphisms $f \in U'$ for which the inductive step follows easily by extending on the whole of $\mathcal{X}$ the time-dependent vector field $\zeta(x,t)$. This is possible by adapting the standard techniques of lifting of stratified vector fields [7, 21, 22, 34, 38]. It was a merit of Andrew du Plessis to discover this key idea of using Mather time-dependent vector fields when he was examiner for my Ph.D. thesis directed by D. Trotman in the summer of 1997. □

Another way to prove the stratified transversality theorem could be the following. In [30], using a theorem of D. McDuff [26] on the classification of the distinguished subgroup of the diffeomorphisms of a compact manifold with boundary, I extend to non-compact manifolds a well known Epstein–Thurston theorem and show that :

Theorem 5.2. *If $S = intM$ is a manifold, diffeomorphic to the interior of a compact manifold with boundary, the image of the exponential map generates $Diff_0(S,S)$. In particular every $f \in Diff_0(S,S)$ can be written as a composition $f = \phi_1^1 \circ \cdots \circ \phi_1^s$ of diffeomorphisms ϕ_1^i which are the time 1 map of the flow ϕ^i of a vector field ζ^i on S.*

Question 5.1. *If $\lim_{x \to \partial S} f = 1_{\partial S}$ can we obtain in theorem 5.2, that $\lim_{x \to \partial S} \zeta^i = 0$, for every $i = 1, \ldots, s$?*

If the answer was yes, then by the usual techniques of stratified lifting of vector fields one would deduce easily a new *Extension Theorem for stratified homeomorphisms*, and without using time-dependent vector fields. This seems to me to depend on an apparently difficult improvement of the McDuff theorem.

Proof. *Outlines of the proof in [31] of Theorem 5.1.* We construct $\Phi = \Phi^k$ by the following steps.

First we prove that :

i) "Every smooth manifold S, admits a finite family of smooth and complete vector fields $v_1(x), \ldots, v_r(x)$ which span T_xS for every $x \in S$ and there exists a sufficiently small open ball $B = B(0, \epsilon) \subseteq \mathbb{R}^r$ such that for every $b = (b_1, \ldots, b_r) \in B$, the vector field $\zeta_b = \sum_i b_i v_i$ is again complete".

Then we prove that :

ii) "The smooth map $G : S \times B \to S$ defined by $G(x, b) = \psi_1^b(x)$, where $\psi^b : S \times \mathbb{R} \to S$ is the flow of ζ_b, satisfies the submersivity of all partial maps $G_x : B \to S$ defined by $G_x(b) = \psi_1^b(x)$".

Finally we use a very nice and ingenious way to apply Sard's Theorem, discovered by R. Abraham [1] (see also the 1987 book [17], p. 51 for comments) to prove that:

iii) "If $j : B \to C^\infty(S, S)$ denotes the map $j(b) = G_b$, then the subset $M = \{b \in B \mid j^0 G_b$ is not transverse to $h_{|h^{-1}(S)} \times g_{|g^{-1}(S)}\}$, where $j^0 G_b$ is the graph of the map $G_b = \psi_1^b$, has measure zero in $\mathbb{R}^r$.

So taking $b \in B - M$ the diffeomorphism $f = G_b = \psi_1^b : S \to S$ is transverse to $h_{|h^{-1}(S)} \times g_{|g^{-1}(S)}$. Thus $f \circ h_{|h^{-1}(S)}$ is transverse to $g_{|g^{-1}(S)}$".

Again, an extension of ζ_b on the whole of $\mathcal{X}$ (to obtain $\Phi_1 = \Phi_1^k$), and the conclusion of the inductive step follow by the usual techniques of lifting of vector fields. $\square$

In this proof (of [31]) we obtain as Goresky hoped in his transversality theorem ([15], 5.3.) a diffeomorphism f which is transversalizing and *simultaneously* the time 1 map ψ_1^b of the flow of a vector field ζ_b <u>not depending on time</u>.

I like also to make precise here that :

(1) : part of this idea *iii*) appears already, in Lemma 6.3.4, at the end of Goresky's thesis, where the author gave the following proposition and comment [14, p. 183] :

Proposition 5.1. *Suppose $\mathcal{W}_1$ and $\mathcal{W}_2$ are Whitney stratified objects in a manifold M and U an open subset of M such that $\mathcal{W}_1 \cap U$ is transverse to $\mathcal{W}_2 \cap U$. Let K be a closed subset of U. Then there is a smooth vector field η on M, $\eta_{|K} \equiv 0$ such that $\Phi_1(\mathcal{W}_1)$ is transverse to $\mathcal{W}_2$ where $\phi_1 : M \to M$ is the diffeomorphism generated at the time 1 by the flow of η.*

Comment. *"Note, however, that if η does not have a compact support then the one-parameter group of diffeomorphisms $\phi_t : M \to M$ do <u>not</u> describe a continuous path in $Diff(M)$ under the C^1-topology. This lemma is, therefore not a suitable substitute for the transversality theorem".*

So part *i)* and *ii)* of our proof in [31] fill this gap and allow one to obtain a complete proof of a more general stratified transversality theorem.

(2) : the idea to use the Abraham method to apply Sard's Thorem, appeared again in a better explained and formalized way in 1987 [17] (many years after the thesis) but also in this case is developed only in an example (*Example 1.3.7.* p. 39) and in the very particular case where the manifold S was the projective space $\mathbb{CP}^n$ which is a compact manifold.

On the other hand, the strata of a stratification are not in general compact manifolds, and moreover for compact manifolds, the properties *i)* and *ii)* become easy to prove.

Finally, using the definitions of Goresky–MacPherson in [17] the joint proofs of the i) and ii) may be stated in a more elegant way as follows :

Theorem 5.3. *Every smooth non compact manifold S admits a submersive family $G = \{G_b\}_{b\in B}$ of self maps.*

I like to recall here that (after we read and re-read Goresky's Ph.D. thesis), it was the merit of David Trotman in the summer 2000, to point-out that this was the *good property* to prove for a non compact manifold (such as the generic stratum of a regular stratification) in order to obtain diffeomorphisms which *at the same time* are transversalizing and lie in a one parameter group (this happened while we were working to answer L. Siebenmann, who asked us if we knew another way to prove the stratified transversality theorem which did not use time-dependent vector fields).

Given the technical and historical difference between our two proofs of Theorem 5.1, which extend (and also clarify) the Goresky transversality theorem, we now look at the main corollaries of this theorem.

Suppose now, as in the *Transversality Lemma* of Goresky, that $\mathcal{W}$ is a substratified object of $\mathcal{X}$, and that the map $h = i : \mathcal{W} \hookrightarrow \mathcal{X}$ is the stratified inclusion of $\mathcal{W}$ in $\mathcal{X}$, and consider the map $h' = \Phi_1 \circ h$.

Because the transversalizing deformation Φ_1 is a stratified homeomorphism, and hence is a diffeomorphism on each stratum, one can easily see that the condition *"$h' = \Phi_1 \circ h$ is transverse to g"* may be reread as *"$\mathcal{W}' = \Phi_1(\mathcal{W})$ is transverse to g"*. Thus we have the following corollary which generalizes the *Transversality Lemma* of Goresky, without the π-fibre condition on the substratified object $\mathcal{W}$ to be deformed.

Corollary 5.1. *Let $\mathcal{X}$ be an abstract stratified set, or a (w)-regular nice stratified subspace of a manifold, and $g : \mathcal{Y} \to \mathcal{X}$ a stratified map defined* *on a stratified space $\mathcal{Y}$.*

Then for each substratified object $\mathcal{W}$ of $\mathcal{X}$ and each open neighbourhood U of W in $\mathcal{X}$, there exists a deformation by isotopy $\mathcal{W}'$ of $\mathcal{W}$ which is transverse to g and such that $W' \subseteq U$. Moreover if C is a closed subset of $\mathcal{X}$ on which $\mathcal{W}$ is transverse to g then we can obtain that $\mathcal{W}' \cap C = \mathcal{W} \cap C$.

Corollary 5.1 holds for stratifications and stratified maps that are more general than those of the *Transversality Lemma* of Goresky [15]. For we do not require either of the two conditions :

i) that g be controlled with respect to two fixed systems of control data $\mathcal{T}_1$ et $\mathcal{T}_2$ respectively of $\mathcal{Y}$ and $\mathcal{X}$ or that g be the restriction of a smooth map $\tilde{g} : M_1 \to M_2$ between two manifolds containing respectively $\mathcal{Y}$ and $\mathcal{X}$;

ii) that $\mathcal{W}$ satisfy the π-fibre condition.

The π-fibre condition (or to be more precise its version stratum by stratum redefined in [29, p. 160]) is a very strong restriction on the geometry of the substratified object $\mathcal{W}$ of $\mathcal{X}$ and ensures that (b)-regularity be preserved as was shown in [15]; possibly other regularity conditions are preserved. For example this is the case for (a)-regularity, but it could also be true for (w)-regularity or (c)-regularity.

In Corollary 5.1, as we do not consider any regularity condition for $\mathcal{W}$ other than being a substratified object of $\mathcal{X}$, the problem of the preservation of such a condition by deformation by isotopy does not arise. We will talk in §5.1. about this delicate problem.

Corollary 5.1 was also used by M. Grinberg, when $g : \mathcal{Y} \hookrightarrow \mathcal{X}$ is the inclusion map and $\dim(Y \cap S) + \dim(W \cap S) < \dim S$ for every stratum S of $\mathcal{X}$, to prove the existence of self-indexing stratified Morse functions on complex algebraic varieties ([18], 2005).

Corollary 5.2. *Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified or a (w)-regular nice stratified subspace of a manifold, and $\mathcal{V}$ a substratified object of $\mathcal{X}$.*

For each substratified object $\mathcal{W}$ of $\mathcal{X}$, and each open neighbourhood U of the support W of $\mathcal{W}$ in A there exists a deformation by isotopy $\mathcal{W}'$ of $\mathcal{W}$, transverse to $\mathcal{V}$ in $\mathcal{X}$, with support $W' \subseteq U$.

If Z is a closed subset of A at each point of which $\mathcal{W}$ is transverse to $\mathcal{V}$ one can obtain moreover that the transversalizing isotopy $\Phi : A \times I \to A$ satisfies $\Phi_{t|Z} = id$ for all $t \in I$ and so $\mathcal{W}' \cap Z = \mathcal{W} \cap Z$.

Remark 5.1. K. Bekka has shown [3] that (c)-regular stratifications admit a system of control data; so both (b)-regular and (c)-regular stratified sets are abstract stratified sets and hence Theorem 5.1 and its corollaries 5.1 and 5.2 hold for them.

Remark 5.2. Because (L)-regular nice stratifications are (w)-regular [38], Theorem 5.1 holds also for Mostowski's (L)-regular stratified spaces.

5.1. *Preservation of regularity after deformation. Open problems*

In general a deformation by isotopy $\mathcal{W}' = \Phi_1(\mathcal{W})$, without supposing the π-fibre condition on $\mathcal{W}$, does not preserve any regularity condition of $\mathcal{W}$ except *"to be an abstract stratified set"*.

In [31] we introduce then the following notion of differentiability :

Definition 5.3. We say that a stratified morphism $f : \mathcal{X} \to \mathcal{X}'$ is *semidifferentiable at* x *of* $X \in \Sigma$ if for each stratum $Y > X$ (i.e. $\overline{Y} \supseteq X$) and for each sequence $\{(y_n, v_n)\}_n$ in the tangent space TY we have that $\lim_{n\to\infty}(y_n, v_n) = (x, v) \in TX$ implies $\lim_{n\to\infty} f_{Y*y_n}(v_n) = f_{X*x}(v)$.

We say f is *semidifferentiable on a stratum* X iff it is semidifferentiable at every $x \in X$ and that f is *semidifferentiable* iff it is semidifferentiable on every stratum $X \in \Sigma$.

Semidifferentiability (at x) is weaker than C^1-differentiability of f (at x) and provides sufficient conditions for a stratified homeomorphism (C^1 diffeomorphism on each stratum) to preserve some regularity of stratified subspaces. In [31] we show :

Theorem 5.4. *Let* $\mathcal{X} = (A, \Sigma)$ *be a* (c)*-regular stratified space, with* A *a closed subset of a* C^∞ *manifold* M *and let* $g : \mathcal{Y} \to \mathcal{X}$ *be a stratified map defined on a stratification* $\mathcal{Y}$.

For each substratified object $\mathcal{W}$ *of* $\mathcal{X}$ *and each open neighbourhood* U *of* W *in* A *there exists a stratified isotopy* $\Phi_t : \mathcal{X} \to \mathcal{X}$ *such that the deformation* $\mathcal{W}' = \Phi_1(\mathcal{W})$ *is transverse to* g, *and* $W' \subseteq U$.

If moreover Φ_1 *is semidifferentiable, then:*

i) $\mathcal{W}$ (c)*-regular* $\Rightarrow$ $\mathcal{W}'$*is* (c)*-regular;*

ii) $\mathcal{W}$ (a)*-regular* $\Rightarrow$ $\mathcal{W}'$ *is* (a)*-regular.*

On the other hand, we do not know currently a sufficient condition for Φ_1 to be semidifferentiable. The use of *continuous* liftings of vector fields in the transversality theorem is a necessary condition [34]. In any case, without assuming the π-fibre condition on $\mathcal{W}$, the following problem remains open:

Problem 5.1. *Can we obtain a stratified transversality theorem in which the deformation by isotopy $\mathcal{W}' = \Phi_1(\mathcal{W})$ transverse to $\mathcal{V}$ preserves some of the regularity conditions of $\mathcal{W}$ such as (a)- or (b)- or (c)- regularity ?*

This problem could be solved if one proves the smooth version of the Whitney fibering conjecture [35, 36] :

Conjecture 5.1. *Every Whitney stratification $\mathcal{X}$, is such that for each point x in a stratum X of $\mathcal{X}$, there exists a neighbourhood U of x in A which admits an l-dimensional stratified foliation $\mathcal{F} = \{F_y\}_y$ (where $l = \dim X$ and F_y denotes the leaf containing y) such that for every $x' \in U \cap X$, $\lim_{y \to x'} T_y F_y = T_{x'} X$.*

5.2. *Stratification of the transverse union and intersection. Open problem*

Let $\mathcal{W}' = (W', \Sigma_{\mathcal{W}'})$ be a deformation of $\mathcal{W} = (W, \Sigma_{\mathcal{W}})$ transverse to $\mathcal{V} = (V, \Sigma_{\mathcal{V}})$ and $\{V_\alpha\}_\alpha$, $\{W'_\beta\}_\beta$ the families of strata of $\mathcal{W}'$ and $\mathcal{V}$.

If $C(H)$ denotes the family of the connected components of a space H, then $V \cup W'$ and $V \cap W'$ have natural partitions in smooth manifolds defined by :

$$\Sigma_{V \cap W'} = \sqcup_{V_\beta \subseteq V\, , W'_\alpha \subseteq W'} \; C(V_\beta \cap W'_\alpha)$$

and respectively by

$$\Sigma_{V \cup W'} = \sqcup_{V_\beta \subseteq V\, , W'_\alpha \subseteq W'} C(V_\beta - W'_\alpha) \; \sqcup C(W'_\alpha - V_\beta) \; \sqcup \; C(V_\beta \cap W'_\alpha)$$

called *transverse intersection* $\mathcal{V} \cap_t \mathcal{W}'$ and *transverse union* $\mathcal{V} \cup_t \mathcal{W}'$ of $\mathcal{V}$ and $\mathcal{W}'$. Unfortunately, as we show in [31] (Examples 3.17 and 3.16), these partitions do not define in general natural stratifications for $V \cap W'$ and $V \cup W'$, for two reasons:

i) in general $\mathcal{V} \cap_t \mathcal{W}'$ and (thus) $\mathcal{V} \cup_t \mathcal{W}'$ are not locally finite ;
ii) in general $\mathcal{V} \cup_t \mathcal{W}'$ does not satisfy the frontier condition.

It follows that if $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a stratified map, not necessarily controlled, and $\mathcal{W}$ a substratified space of $\mathcal{X}_2$, not necessarily π-fibre, in general it is not true that $f^{-1}(\mathcal{W})$ is a substratified space of $\mathcal{X}_1$. So the good morphisms for Whitney cohomology WH^* have to be controlled maps.

It also follows that we cannot define a *transverse sum operation* in the set $WH_k(\mathcal{X})$ when $\mathcal{X}$ is an arbitrary Whitney stratification (not a manifold). This is the main raison for which Whitney homology WH_* (whose cycles are not defined as π-fibres) is a theory less rich in geometric interpretations than Whitney cohomology WH^*. This obstruction could may be overcome by answering the following question :

Problem 5.2. *Is it possible to find* <u>*two*</u> *deformations by isotopy* $\mathcal{V}' = \Phi_1(\mathcal{V})$ *and* $\mathcal{W}' = \Psi_1(\mathcal{W})$ *such that* $\mathcal{V}'$ *is transverse to* $\mathcal{W}'$ *and moreover* $\mathcal{V}' \cap_t \mathcal{W}'$ *and* $\mathcal{V}' \cup_t \mathcal{W}'$ *are locally finite and satisfy the frontier condition, defining so two stratifications ?*

A positive answer to this question would give the possibility to structure the *homology* set $WH_k(\mathcal{X})$ as an abelian group with the *transverse sum operation* when $\mathcal{X}$ is an arbitrary Whitney stratification, and this group structure could be a powerful tool to approach in an algebraic way the Goresky conjecture (§3, Conjecture 3.1).

For a manifold transverse to all strata of an analytic stratification Σ such that for each stratum $S \in \Sigma$, $\overline{S}$ and $\overline{S} - S$ are analytic sets in 1972 D. Chéniot [6], without assuming any regularity condition for Σ, proved that :

Theorem 5.5. *Let* V *be a complex analytic set, in an open set* U *of* $\mathbb{C}^n$*, equipped with an analytic stratification* Σ*. Let* M *be a complex analytic submanifold of* U*, transverse to every stratum of* Σ*.*

Then the trace $\Sigma_{M \cap V} = \{S \cap M \mid S \in \Sigma\}$ *of* M *over* Σ *is a stratification (so locally finite and satisfying the frontier condition).*

Here the stratifications are intended by considering the strata to be not necessarily connected, with so some difference with respect to our definitions. In such a complex analytic context the essential property used was that *"for each stratum* S *of* V*, one has* $\overline{S \cap M} = \overline{S} \cap M$*"* ; D. Chéniot also proves that *this key property is a sufficient hypothesis to obtain the conclusion of theorem 5.5 also for* V *and* M *real analytic* [6]. This key property is always satisfied when Σ is a real or complex Whitney stratification.

6. Transverse intersections and other applications

The problem of knowing when a transverse intersection and a transverse union is a *good* stratification also arises for knowing when $\mathcal{W} \cap_t \mathcal{V}$ and $\mathcal{W} \cup_t \mathcal{V}$ preserve the regularity condition of $\mathcal{W}$ and $\mathcal{V}$. This becomes already very useful in the simpler case when the stratification of the ambient space $\mathcal{X}$ and/or one of the two subspaces (as an example $\mathcal{V}$) reduce to two manifolds.

In 1976 C. G. Gibson [12] proved that Whitney (b)-regularity is preserved by transverse intersection in a manifold :

Theorem 6.1. *Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be Whitney stratifications in a smooth manifold M. If $\mathcal{V}_1, \ldots, \mathcal{V}_n$ are in general position then $\mathcal{V}_1 \cap \cdots \cap \mathcal{V}_n$ is a Whitney stratification.*

More recently in 2002, P. Orro and D. Trotman [37] gave a unique proof of the corresponding theorem holding for many regularity conditions (although for the $(a+\delta)$ regularity the result was first proved in 2000, [5]):

Theorem 6.2. *The regularity conditions (a), (b), (w), $(a+\delta)$, $(a+r^e)$ for every $e \in [0,1[$, are invariant by transverse intersection in a manifold of two stratifications with C^2 strata.*

This result was essential in proving the main theorem of [37] :

Theorem 6.3. *Let $\mathcal{Z}$ be a closed set in a manifold, stratified by C^k strata, $k \geq 2$ and $(a+r^e)$-regular, with $e \in [0,1[$, relatively to a stratum Y.*

For every $y \in Y$, the fibre $(C_Y \mathcal{Z})_y$ of the normal cone $C_Y \mathcal{Z}$, coincides with the fibre $C_y(\mathcal{Z}_y)$ of the tangent cone to the fibre $\mathcal{Z}_y = \mathcal{Z} \cap \pi^{-1}(y)$ of a C^1 retraction π over Y.

6.1. *Application to abstract stratified and (c)-regular homology*

For (c)-regular stratifications, in 1991 K. Bekka [3] proved the following very useful propositions :

Proposition 6.1. *Let $f : M \to N$ be a C^1 map between C^1-manifolds and let $\mathcal{W} \subseteq N$ a (c)-regular stratified space. If f is transverse to $\mathcal{W}$ then $f^{-1}(\mathcal{W})$ defines a (c)-regular stratified subspace of M.*

Proposition 6.2. *Let M be a C^1-manifold and let $\mathcal{V}, \mathcal{W} \subseteq M$ be two (c)-regular stratified spaces.*

If $\mathcal{V}$ and $\mathcal{W}$ are transverse in M then the transverse intersection $\mathcal{V} \cap_t \mathcal{W}$ and the transverse union $\mathcal{V} \cup_t \mathcal{W}$ are again two (c)-regular stratified spaces.

Proposition 6.3. *Let $f : M \to N$ be a C^1 map between C^1-manifolds and $\mathcal{V} \subseteq M$ and $\mathcal{W} \subseteq N$ two (c)-regular stratified spaces.*

If f sends $\mathcal{V}$ transversally on $\mathcal{W}$ then $f^{-1}_{|\mathcal{V}}(\mathcal{W}) = \mathcal{V} \cap_t f^{-1}(\mathcal{W})$ is a (c)-regular stratified space.

For abstract stratified sets, the proofs of the corresponding statements, already obtained by Goresky in his thesis [14], are again true with similar proofs. Thanks to these results (and following the ideas of Goresky in [15]) one can construct new homology and cohomology theories in which the ambient space $\mathcal{X}$ and its cycles $\mathcal{V}$ and cocycles $\mathcal{W}$ are Thom–Mather abstract stratified sets (see also [14]) and/or Bekka (c)-regular stratifications (instead of Whitney (b)-regular stratifications [15]).

One obtains so the new theories AH_*, AH^* and BH_*, BH^* [4, 29] (instead of WH_*, WH^* of [15]). In these new theories, Theorem 5.1 with its corollaries 5.1 and 5.2, play again the role of the fundamental transversality theorem, necessary to give a geometrical meaning to the homology and cohomology algebraic operations. In this way, *all* geometric results of §3 are again true for the abstract stratified homology AH, and most of them also hold for the (c)-regular homology BH [4], [29, Chapter IV].

6.2. *Some applications to homotopy of stratified spaces*

In 1999 C. Eyral used stratified transversality to study the homotopy of stratified spaces [8]. First he proved the following theorem in which the stratified transversality is a necessary hypothesis (see [8] for a counterexample):

Theorem 6.4. *Let M be a C^1 manifold of dimension n, Y a closed subset of M equipped with a Whitney stratification Σ of dimension d and let N be a submanifold of M transverse to each stratum of Σ.*

Then the pair $(N, N-(N\cap Y))$ is $(n-1-d)$-connected.

and by which one deduces immediately the familiar results :

Corollary 6.1. *Let M be a real analytic manifold of dimension n and Y a closed real analytic subspace of M of dimension d. Then the pair $(M, M-Y)$ is $(n-1-d)$-connected.*

Corollary 6.2. *Let M be a C^1 manifold and N a closed submanifold of M of codimension c. Then the pair $(M, M-N)$ is $(c-1)$-connected.*

Then, for a compact and real analytic ambient manifold M, Eyral also improves Theorem 6.4 by proving that it remains true when one considers (instead of a submanifold N) a Whitney stratification Σ' of a closed subset X of M, transverse to Σ in M.

Theorem 6.5. *Let M be a compact real analytic manifold of dimension n, Y, X two closed real analytic subspaces of M equipped with two Whitney stratifications Σ and Σ' transverse in M. Then the pair $(X, X-(X\cap Y))$ is $(n-1-d)$-connected (where $d=\dim\Sigma$).*

Theorems 6.4 and 6.5 are useful in studying the *global rectified homotopical depth* (a notion introduced in 1997 by Eyral in his Ph.D. thesis by reconsidering the *Grothendieck rectified homotopical depth*, see [9]) to prove a theorem of Lefschetz type for quasi-projective singular varieties which extends previous theorems of Goresky–MacPherson ([17, Theorem II.5.2] and Hamm–Lê ([19 Theorem 2.1.4].

Eyral conjectures finally that Theorem 6.5 could also be true without the hypotheses of compactness and analyticity on M,X,Y.

7. More on Goresky's stratified transversality. Supertransversality

In chapter I of his thesis [14], M. Goresky gives the following definitions:

Definition 7.1. Let $f : M \to N$ be a C^∞ map, and $\mathcal{W}_1 \subseteq M$ and $\mathcal{W}_2 \subseteq N$ two closed Whitney substratified objects. One says that f *takes* $\mathcal{W}_1$ *transversally to* $\mathcal{W}_2$ *(on a closed subset* $K \subseteq N$*)* if the stratified map $f_{|\mathcal{W}_1} : \mathcal{W}_1 \to N$ is transverse to $\mathcal{W}_2$ (on K).

Of course if $f \in Diff(M, M)$ this means that $f(\mathcal{W}_1)$ is transverse to $\mathcal{W}_2$.

A map $f : M \to N$ is said to *take* $\mathcal{W}_1$ *supertransversally to* $\mathcal{W}_2$ if for every pair of strata $A \subseteq \mathcal{W}_1$ and $B \subseteq \mathcal{W}_2$, f takes A transversally to B and moreover for every $p \in \overline{A} \subseteq M$ and τ_1 limit of tangent planes of A and whenever $q = f(p) \in \overline{B} \subseteq N$ and τ_2 is limit of tangent planes of B, then $f_{*p}(\tau_1) + \tau_2 = T_qN$.

Clearly if $\mathcal{W}_1 = A$ and $\mathcal{W}_2 = B$ are closed manifolds, f takes $\mathcal{W}_1$ supertransversally to $\mathcal{W}_2$ if and only if f takes $\mathcal{W}_1$ transversally to $\mathcal{W}_2$.

Moreover, if $\mathcal{W}_1 = A$ and $\mathcal{W}_2 = B$ are *two closed* Whitney objects, f takes $\mathcal{W}_1$ supertransversally to $\mathcal{W}_2$ if and only if f takes $\mathcal{W}_1$ transversally to $\mathcal{W}_2$. This follows by the (a)-regularity of $\mathcal{W}_1$ and $\mathcal{W}_2$ since if $A \subseteq M$ and $B \subseteq N$ are strata of $\mathcal{W}_1$ and $\mathcal{W}_2$, and (with the above notations), $p \in \overline{A}$, $q = f(p) \in \overline{B}$, by closedness there exist two strata $A' \leq A$ and $B' \leq B$ for which $p \in A'$, $q \in B'$ and such that by hypothesis $f_{*p}(T_pA') + T_qB' = T_qN$. So by the (a)-regularity of $A \leq A'$ and $B \leq B'$, $\tau_1 \supseteq T_pA'$ and $\tau_2 \supseteq T_qB'$ and one finds the supertransversality $f_{*p}(\tau_1) + \tau_2 = T_qN$ [14, 1.3.1].

Goresky introduced this new notion of supertransversality with the project to prove the following theorem [14, 1.2.2] :

Theorem 7.1. *Suppose* M, N *are manifolds,* K *is a closed subset of* N*,* $\mathcal{W}_1 \subseteq M$ *and* $\mathcal{W}_2 \subseteq N$ *are closed Whitney stratified objects. Suppose*

$f{:}M \to N$ is a smooth map, which takes $\mathcal{W}_1$ transversally to $\mathcal{W}_2$ on K. Then:

i) There is a neighborhood U of f in $C^\infty(M,N)$ so that if $g \in U$, then g takes $\mathcal{W}_1$ transversally to $\mathcal{W}_2$ on K.

ii) For any neighborhood U' of f in $C^\infty(M,N)$ there exists a map $f' \in U'$ which takes $\mathcal{W}_1$ transversally to $\mathcal{W}_2$ on N, and satisfies $f'_{|f^{-1}(K)} = f_{|f^{-1}(K)}$.

Furthermore f' may be chosen to be homotopic to f by a smooth homotopy which is constant on K.

The techniques used by Goresky in his thesis [14] to find a proof of Theorem 7.1 can be resumed in some utilisations of the *families of geodesics* of Mather [20] to construct isotopies sufficiently close to the identity (1.3.3 and 1.3.4), some lemmas (1.3.1, 1.3.2, 1.3.3) and discussions on supertransversality and the proposition (1.3.4) below :

Proposition 7.1. *Suppose $f \in C^\infty(M,N)$, A a closed submanifold of M and B a submanifold of N. Let $K \subseteq N$ be a closed subset and U be a C^1 neighbourhood of f in $C^\infty(M,N)$. Suppose f takes A transversally to B on K and let $K' = f^{-1}(K)$. Then there exists a $g \in U$ such that $g_{|K'} = f_{|K'}$ and g takes A transversally to B.*

I consider important to present it because :

i) this was historically the original project of a stratified transversality theorem by Goresky ;

ii) the fact that Theorem 7.1 above or (1.2.2) in [14] remained unpublished since 1976, induced probably Goresky to find a new formulation of it, in the completely revised version of 1981 [15].

Finally, Goresky ended Chapter I of his thesis [14], by deducing from it corollary 7.1 below and by adding the following proposition (already announced in §6.1) essential to give geometrical meaning to the operation involving the Whitney stratified cycles and cocycles of his geometric homology and cohomology theory :

Corollary 7.1. *If $\mathcal{W}_1$ and $\mathcal{W}_2$ are closed Whitney stratified objects in a manifold M which are transverse on a closed subset $K \subseteq M$, then there exists a diffeomorphism $\phi : M \to M$ arbitrarily close to 1_M in the C^1 topology, so that $\phi(\mathcal{W}_1)$ is transverse to $\mathcal{W}_2$ and $\phi_{|K} = 1_K$.*

Proposition 7.2. *Suppose $f : M \to N$ is a smooth map between smooth manifolds and suppose $\mathcal{W}_1 \subseteq M$ and $\mathcal{W}_2 \subseteq N$ are Whitney stratified objects.*

Suppose f takes $\mathcal{W}_1$ transversally to $\mathcal{W}_2$, then $\mathcal{W}_1 \cap f^{-1}(\mathcal{W}_2)$ is a Whitney stratified object.

References

1. R. Abraham, *Transversality in manifold of mapping*, Bull. Amer. Math. Soc. 143 (1969), 470-474.
2. E. Akin, *Manifold phenomena in the theory of polyedra*, Trans. Amer. Math. Soc. 143 (1969), 413-473.
3. K. Bekka, *C-régularité et trivialité topologique*, Singularity theory and its applications, Warwick 1989, Part I, Lecture Notes in Math. **1462**, Springer, Berlin (1991) 42-62.
4. K. Bekka et C. Murolo, *Homologie d'espaces stratifiés*, C. R. Acad. Sci. Paris, t. 331, Série I (2000) 703-708.
5. K. Bekka and D. Trotman, *Weakly Whitney stratified sets*, Real and complex singularities, Sao Carlos 1998 (edited by J.W. Bruce and F. Tari), Chapman and Hall/CRC Research Notes in Mathematics 412, Boca Raton, Florida (2000), 1-15.
6. D. Chéniot, *Sur les sections transversales d'un ensemble stratifié*, C. R. Acad. Sci., t. 275, série A, (1972), 915-916.
7. A. A. du Plessis, *Continuous controlled vector fields*, Singularity theory (Liverpool, 1996, edited by J. W. Bruce and D. M. Q. Mond), London Math. Soc. Lecture Notes **263**, Cambridge Univ. Press, Cambridge, (1999), 189-197.
8. C. Eyral, *Sur l'homotopie des espaces stratifiés*, International mathematics research Notices (1999), N. 13, 717-734.
9. C. Eyral, *Tomographie des variétés singulières et théorèmes de Lefschetz*, Proc. London Math. Soc. (3) 83 (2001) 141-175.
10. C. Freifeld, *One parameter subgroups do not fill a neighbourhood of the identity in an infinite dimensional Lie (pseudo-)group*, Battelle Rencontre, Lectures in Mathematics and Physics, Benjamin, New-York (1967), 538-543.
11. W. Fulton, *Intersection Theory*, Springer Verlag, Berlin-Heidelberg (1984).
12. C. G. Gibson, K. Wirthmüller, A. A. du Plessis and E. J. N. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Math. **552**, Springer–Verlag (1976).
13. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics 14, Springer–Verlag, Berlin and New York (1973).
14. M. Goresky, *Geometric Cohomology and homology of stratified objects*, Ph.D. thesis, Brown University (1976).
15. M. Goresky, *Whitney stratified chains and cochains*, Trans. Amer. Math. Soc. **267** (1981), 175-196.
16. M. Goresky and R. MacPherson, *Intersection homology*, Topology **19** (1980), 135-162.
17. M. Goresky and R. MacPherson, *Stratified Morse theory*, Springer–Verlag, Berlin (1987).
18. M. Grinberg, *Gradient-Like Flows and Self-Indexing in Stratified Morse Theory*, Topology 44 (2005), 175-202.
19. H.A. Hamm and D.T. Lê, *Lefschetz theorems on quasi-projective varieties*, Bull. Soc. Math. France 113 (1985), 123-142.
20. J. Mather, *Stability of C^∞ mappings: II. Infinitesimal stability implies stability*, Annals of Mathematics **89** (1969), 254-291.
21. J. Mather, *Notes on topological stability*, Mimeographed notes, Harvard University (1970).
22. J. Mather, *Stratifications and mappings*, Dynamical Systems (M. Peixoto, Editor), Academic Press, New York (1971), 195-223.
23. C. McCrory, *Poincaré duality in spaces with singularities*, Ph.D. thesis, Brandeis University (1972).

24. C. McCrory, *Stratified general position*, Algebraic and Geometric Topology, Santa Barbara 1977, Lecture Notes in Math. **664**, Springer–Verlag, Berlin and New York (1978), 142-146.
25. J. Milnor, *Remarks on Infinite-dimensional Lie groups*, Relativity, Groups and Topology II. Les Houches Session XL, 1983. B.S. de Witt & R. Stora Editors. NorthHolland. Amsterdam (1984).
26. D. McDuff, *The lattice of normal subgroups of the group of diffeomorphisms or homeomorphisms of an open manifold*, J. London Math. Soc. (2), 18 (1978), 353-364.
27. C. Murolo, *Whitney homology, cohomology and Steenrod squares*, Ricerche di Matematica **43** (1994), 175-204.
28. C. Murolo, *The Steenrod p-powers in Whitney cohomology*, Topology and its Applications **68**, (1996), 133-151.
29. C. Murolo, *Semidifférentiabilité, transversalité et homologie de stratifications régulières*, Ph.D. thesis, Université de Provence, Marseille (1997).
30. C. Murolo *Diffeomorphisms lying in one-parameter groups and extension of stratified homeomorphisms* Math. Proc. Cambridge Philos. Soc. 130 (2001), no. 2, 333–341.
31. C. Murolo, D. Trotman, A. du Plessis, *Stratified Transversality by Isotopy*, Trans. Amer. Math. Soc. 355 (2003), n. 12, 4881-4900.
32. C. Murolo, A. du Plessis, D. Trotman, *Stratified Transversality by Isotopy via time-dependent vector fields*, J. London Math. Soc. (2) 71 (2005), 516-530
33. C. Murolo and D. Trotman, *Semidifferentiable stratified morphisms*, C. R. Acad. Sci. Paris, t 329, Série I, 147-152 (1999).
34. C. Murolo and D. Trotman, *Relèvements continus de champs de vecteurs*, Bull. Sci. Math., 125, 4 (2001), 253-278.
35. C. Murolo and D. Trotman, *Horizontally-C^1 controlled stratified maps and Thom's first isotopy theorem*, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), n. 8, 707–712.
36. C. Murolo and D. Trotman, *Semidifférentiabilité de Morphismes Stratifiés et Version Lisse de la conjecture de fibration de Whitney*, Proceedings of 12th MSJ-IRI symposium "Singularity Theory and its Applications", to appear 2007.
37. P. Orro and D. Trotman, *Cône normal et régularité de Kuo-Verdier*, Bull. Soc. Math. France, **130** (1) (2002), 71-85.
38. A. Parusiński, *Lipschitz stratifications*, Global Analysis in Modern Mathematics (K. Uhlenbeck, ed.), Proceedings of a Symposium in Honor of Richard Palais' Sixtieth Birthday, Publish or Perish, Houston (1993), 73-91.
39. R. Thom, *Ensembles et morphismes stratifiés*, Bull.A.M.S. **75** (1969), 240-284.
40. D. J. A. Trotman, *Geometric versions of Whitney regularity*, Annales Scientifiques de l'Ecole Normale Supérieure, 4*eme* série, **t. 12**, (1979), 453-463.
41. J.-L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard*, Inventiones Math. **36** (1976), 295-312.
42. H. Whitney, *Local properties of analytic varieties*, Differential and Combinatorial Topology, Princeton Univ. Press, (1965), 205-244.