Principal kinematic formulas for germs of closed definable sets

Nicolas Dutertre (Angers)

Singularity Theory and Regular Stratifications

Summary

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The search for kinematic formulas is one of the main goal of integral geometry. Such formulas have been proved in various contexts by various authors, for instance:

- For convex bodies by Blaschke and Hadwiger;
- For manifolds by Chern and manifolds with boundary by Santaló;
- For PL-sets by Cheeger, Müller and Schrader;
- For sets with positive reach by Federer;
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The Lipschitz-Killing measures

Let $X \subset \mathbb{R}^n$ be a compact definable set equipped with a finite definable Whitney stratification $\mathcal{S} = \{S_a\}_{a \in A}$.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 definable function.

Definition

A point $p \in X$ is a critical point of $f_{|X}$ if it is a critical point of $f_{|S(p)}$, where $S(p)$ is the stratum that contains p.

Definition

If p is an isolated critical point of $f_{|X}$,

$$\mathrm{ind}(f, X, p) = 1 - \chi\Big(X \cap \{f = f(p) - \delta\} \cap B_\epsilon(p)\Big),$$

where $0 < \delta \ll \epsilon \ll 1$.

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Lemma

There exists a definable set $\Gamma_1(X) \subset S^{n-1}$, $dim\Gamma_1(X) < n-1$, such that for $v \notin \Gamma_1(X)$, the function $v^*_{|X}$ has a finite number of critical points ($v^*(y) = \langle v, y \rangle$).

Definition (Gauss-Bonnet measure)

Let $U \subset X$ be a Borel subset. We set

$$\Lambda_0(X, U) = \frac{1}{s_{n-1}} \int_{\mathbb{S}^{n-1}} \sum_{x \in U} ind(v^*, X, x) dv,$$

where $ind(v^*, X, x) = 0$ if x is not a critical point of $v^*_{|X}$.

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Let $U \subset X$ be a Borel subset. For $k \in \{0, \ldots, n-1\}$, we set

$$\Lambda_{n-k}(X, U) = c(n, k) \int_{A_n^k} \Lambda_0(X \cap E, X \cap E \cap U) dE,$$

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1. A_n^k is the affine Grassmannian of affine spaces of dimension k in $\mathbb{R}^n$,
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Remark

1. *For any Borel set U of X, we have*

$$\Lambda_{d+1}(X,U)=\cdots=\Lambda_n(X,U)=0,$$

and $\Lambda_d(X,U)=\mathcal{H}_d(U)$, where d is the dimension of X and $\mathcal{H}_d$ is the d-th dimensional Hausdorff measure in $\mathbb{R}^n$.

2. *If X is smooth then for $k\in\{0,\ldots,d\}$, $\Lambda_k(X,U)$ is equal to*

$$\frac{1}{s_{n-k-1}}\int_U K_{d-k}(x)dx,$$

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The Lipschitz-Killing measures satisfies the following kinematic formula:

Theorem (due to Fu, Bernig-Broecker-Kuppe)

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ be two compact definable sets and let $U \subset X$ and $V \subset Y$ be two Borel sets. For $k \in \{0, \ldots, n\}$, the following kinematic formula holds:

$$\int_{SO(n)\ltimes\mathbb{R}^n} \Lambda_k(X \cap gY, U \cap gV) d\gamma dx$$

$$= \sum_{p+q=k+n} e(p,q,n)\Lambda_p(X,U)\Lambda_q(Y,V),$$

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For $k = 0$, the above formula is called the principal kinematic formula.

Corollary (Principal kinematic formula)

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ be two compact definable sets. We have

$$\int_{SO(n)\ltimes\mathbb{R}^n} \chi(X \cap gY) d\gamma dx = \sum_{p+q=n} e(p,q,n)\Lambda_p(X,X)\Lambda_q(Y,Y).$$

Goal : find a similar formula for germs of closed definable sets.

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Known formulas for germs

The polar invariants were defined by Comte and Merle. They are real versions of the local vanishing Euler characteristics (Brylinski-Dubson-Kashiwara, Lê-Teissier).

Definition

(not the original one) Let $(X,0) \subset (\mathbb{R}^n, 0)$ be a germ of closed definable set. For $k \in \{0, \ldots, n\}$, we set

$$\sigma_k(X,0) = \frac{1}{g_n^{n-k}} \int_{G_n^{n-k}} \left(\frac{1}{s_{n-1}} \int_{\mathbb{S}^{n-1}} \lim_{\epsilon \to 0} \lim_{\delta \to 0^+} \chi\left(X \cap (H + \delta v) \cap \mathbb{B}_\epsilon^n\right) dv \right) dH.$$

Remark

If $d = \dim X$ and d_0 is the dimension of the stratum that contains 0, then $\sigma_0(X,0) = \ldots = \sigma_{d_0}(X,0) = 1$, and $\sigma_{d+1}(X,0) = \ldots = \sigma_n(X,0) = 0$.

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For any germ $(X,0) \subset (\mathbb{R}^n, 0)$ of closed definable set, we have

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Then Comte and Merle proved the following real version of a result due to Lê and Teissier for complex analytic sets (improved later by Nguyen and Valette).

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Let $(X,0) \subset (\mathbb{R}^n,0)$ be a germ closed definable set. For $k \in \{0,\ldots,n-1\}$, we have

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$$\Lambda_k^{\lim}(X,0) := \lim_{\epsilon \to 0} \frac{\Lambda_k(X, X \cap \mathbb{B}_\epsilon^n)}{b_k \epsilon^k}.$$

Note that $\Lambda_k^{\text{loc}}(X,0) \neq \Lambda_k^{\lim}(X,0)$. If $X = \mathbb{R}^2$ then

$\Lambda_0^{\text{loc}}(X,0) = 1,\ \Lambda_1^{\text{loc}}(X,0) = \frac{\pi}{2}$ and $\Lambda_0^{\lim}(X,0) = 0,\ \Lambda_1^{\lim}(X,0) = 0.$

Theorem (D., 2015)

Let $(X,0) \subset (\mathbb{R}^n,0)$ be a germ closed definable set. For $k \in \{0,\ldots,n-1\}$, we have

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The principal kinematic formula for germs

Question: can we replace the $(n-k)$-plane H with any germ of closed definable set ?

Theorem (D., 2020)

Let $(X,0) \subset (\mathbb{R}^n,0)$ and $(Y,0) \subset (\mathbb{R}^n,0)$ be two germs of closed definable sets. The following principal kinematic formula holds:

$$\frac{1}{s_{n-1}^2} \int_{SO(n)\times\mathbb{S}^{n-1}} \lim_{\epsilon\to 0} \lim_{\delta\to 0^+} \chi\left(X \cap (\gamma Y + \delta v) \cap \mathbb{B}_\epsilon^n\right) d\gamma dv$$

$$= \sum_{i=0}^{n} \Lambda_i^{\lim}(X,0) \cdot \sigma_{n-i}(Y,0).$$

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Let us specify this kinematic formula when $d + e = n$, $d = \dim X$ and $e = \dim Y$. We denote by X^d (resp. Y^e) the union of the top-dimensional strata of X (resp. Y). The following corollary is a generalization of the Cauchy-Crofton formula for the density due to Comte.

Corollary

The following formula holds:

$$\frac{1}{s_{n-1}^2} \int_{SO(n)\times\mathbb{S}^{n-1}} \lim_{\epsilon\to 0} \lim_{\delta\to 0^+} \#\left(X^d \cap (\gamma Y^e + \delta v) \cap \mathbb{B}_\epsilon^n\right) d\gamma dv$$

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Let $X \subset \mathbb{S}^{n-1}$ be a compact definable set and let $Y \subset \mathbb{S}^{n-1}$ be a definable set.

Proposition

The following kinematic formula holds:

$$\frac{1}{s_{n-1}} \int_{SO(n)} \chi_c(X \cap \gamma Y) d\gamma = \sum_{i=0}^{n-1} \frac{\tilde{\Lambda}_i(X,X)}{s_i} \cdot \frac{1}{g_n^{i+1}} \int_{G_n^{i+1}} \chi_c(Y \cap H) dH,$$

where the $\tilde{\Lambda}_i$'s, $i = 0, \ldots, n-1$, denote the spherical Lipschitz-Killing measures.

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Idea of the proof. If Y is compact, then it is an application of the generalized spherical Gauss-Bonnet formula and the generalized spherical kinematic formula (due to Fu, Bernig-Broecker-Kuppe).

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Second step: a kinematic formula in the unit ball

Let $X \subset \mathbb{R}^n$ be a closed conic definable set. Let $Y \subset \mathbb{B}^n$ be another definable set.

Proposition

The following kinematic formula holds:

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Idea of the proof. Let us assume first that $0 \notin Y$ and let ϕ be the following definable mapping:

$$\begin{array}{cccc} \phi : & \mathbb{B}^n \setminus \{0\} & \rightarrow & \mathbb{S}^{n-1} \\ & x & \mapsto & \frac{x}{|x|}. \end{array}$$

Then the result follows from:

- Hardt's theorem applied to ϕ,
- the previous spherical kinematic formula,
- the equality: $\frac{\Lambda_k(X, X\cap\mathbb{B}^n)}{b_k} = \frac{\tilde{\Lambda}_{k-1}(X\cap\mathbb{S}^{n-1}, X\cap\mathbb{S}^{n-1})}{s_{k-1}}$ (D., 2012).

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If $0 \in Y$, we denote by $Y^* = Y \setminus \{0\}$. We apply the previous case to Y^* and we use the following equalities:

- $\chi_c(X \cap \gamma Y) = \chi_c(X \cap \gamma Y^*) + 1$ and $\chi_c(Y \cap H) = \chi_c(Y^* \cap H) + 1$,
- $1 - \sum_{i=1}^{n} \frac{\Lambda_i(X, X \cap \mathbb{B}^n)}{b_i} = \Lambda_0(X, X \cap \mathbb{B}^n)$. (D., 2012).

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Proposition

Let $X, Y \subset \mathbb{R}^n$ be two closed conic definable sets. Then

$$\frac{1}{s_{n-1}^2} \int_{SO(n)\times\mathbb{S}^{n-1}} \lim_{\delta\to 0^+} \chi\left(X \cap (\gamma Y + \delta v) \cap \mathbb{B}^n\right) d\gamma dv$$
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Applying the previous proposition to $X \cap \mathbb{B}^n$ and $(Y + \delta v) \cap \mathbb{B}^n$, we get that

$$\frac{1}{s_{n-1}^2} \int_{SO(n)\times\mathbb{S}^{n-1}} \chi\left(X \cap \gamma(Y+\delta v) \cap \mathbb{B}^n\right) dv d\gamma$$
$$= \sum_{i=0}^{n} \frac{\Lambda_i(X, X\cap\mathbb{B}^n)}{b_i} \cdot \frac{1}{g_n^i} \int_{G_n^i} \frac{1}{s_{n-1}} \int_{\mathbb{S}^{n-1}} \chi\left((Y+\delta v)\cap\mathbb{B}^n \cap H\right) dv dH.$$

Passing to the limit as $\delta \to 0^+$ and using Lebesgue's theorem, we obtain that

$$\frac{1}{s_{n-1}^2} \int_{SO(n)\times\mathbb{S}^{n-1}} \lim_{\delta\to 0^+} \chi\left(X \cap (\gamma Y+\delta v) \cap \mathbb{B}^n\right) dv d\gamma$$
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We conclude with the following observation:

$$\lim_{\delta \to 0^+} \chi\left((Y + \delta v) \cap \mathbb{B}^n \cap H\right) = \lim_{\delta \to 0^+} \chi\left(Y \cap \mathbb{B}^n \cap (H - \delta v)\right),$$

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The following kinematic formula is the corollary of the previous one and the Gauss-Bonnet formula for the real Milnor fibre (D., 2015).

Theorem

Let $(X,0) \subset (\mathbb{R}^n,0)$ and $(Y,0) \subset (\mathbb{R}^n,0)$ be two germs of closed definable sets. The following principal kinematic formula holds:

$$\frac{1}{s_{n-1}^2} \int_{SO(n)\times\mathbb{S}^{n-1}} \lim_{\epsilon\to 0} \lim_{\delta\to 0^+} \Lambda_0\left(X \cap (\gamma Y + \delta v), X \cap (\gamma Y + \delta v) \cap \mathbb{B}_\epsilon^n\right) d\gamma dv$$

$$= \sum_{i=0}^{n} \Lambda_i^{\lim}(X,0) \cdot \Lambda_{n-i}^{\lim}(Y,0).$$

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