Proof of Whitney fibering conjecture

joint work with Adam Parusiński

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Equisingularity Problems

- Given an algebraic family of (real or complex) algebraic sets $X_t, t \in T$, is this family generically equisingular?
- Given an analytic family of analytic set germs $X_t, t \in T$, is this family generically equisingular?
- Given an analytic family of analytic function germs $f_t, t \in T$, is this family generically equisingular?
- etc.

i. e. $\exists Z \subset T$, $\dim Z < \dim T$, $X_t \sim X_{t'}$ if t, t' are in the same c.c. of $T \setminus Z$.

The answer depends on what we mean by "$\sim$";

- Yes for topological equisingularity;
- No for differentiable equsingularity;
- Yes/No for lipschitz equisingulartiy.

Examples

- Family of plane curves

$$y^2 - t^3x^2 - x^3 = 0$$

- Family of germs $(X_t, 0) \subset (\mathbb{R}^2, 0)$

$$y(y - x)(y + x)(y - tx) = 0$$

is not C^1 generically trivial. [Whitney, 1965]

- Family of function germs

$$f_t(x, y) = y^3 - 3tx^4y + 2x^6$$

is not bi-lipschitz generically trivial. [Parusiński, Henry]

How topological trivializations can be constructed.

- By means of resolution of singularities (blow-analytic equivalence, Kuo);
  (+) good properties of trivializations;
  (−) known only for families of isolated singularities.

- Stratification Theory (Whitney, Thom, Mather, Kuo, Verdier, Trotman...);
  (+) geometric construction, coordinate free;
  (−) Hard to understerstand trivializations (flows of controlled vector fields).

- Algebro-geometric equisingularity of Zariski (Zariski, Varchenko);
  (−) Depends on the system of coordinates;
  (+) constructive and even algorithmic (based on taking subsequent discriminants as in CAD);
  (+) very good properties of trivializations (Parusiński,-, this talk).

Canonical Stratification associated to a system of pseudoploynomials.

- Consider a *system of pseudopolynomials* in $x = (x_1, ..., x_n) \in \mathbb{K}^n$

$$F_i(x_1, \ldots, x_i) = x_i^{d_i} + \sum_{j=1}^{d_i} A_{i-1,j}(x_1, \ldots, x_{i-1}) x_i^{d_i - j}, \quad i = 1, \ldots, n, \qquad (1)$$

with $\mathbb{K}$-analytic coefficients $A_{i-1,j}$, satisfying

- 1. there are $\varepsilon_j > 0$, $j = 1, \ldots, n$, such that F_i are defined on

$$U_i = \prod_{j=1}^{i} D_j$$

  where $D_j = \{|x_j| < \varepsilon_j\}$.

  2. F_i does not vanish on $U_{i-1} \times \partial D_i$, where $\partial D_i = \{|x_i| = \varepsilon_i\}$.
  3. for every i, the discriminant of $F_{i,red}$ divides F_{i-1}.

- It may happen that $d_i = 0$. Then $F_i \equiv 1$ and we set by convention $F_j \equiv 1$ for $j < i$.

For $i < k$ we denote by $\pi_{k,i} : U_k \to U_i$ the standard projection.

- For each i we define a filtration

$$U_i = X_i^i \supset X_{i-1}^i \supset \cdots \supset X_0^i, \tag{2}$$

  where

- 1. $X_0^1 = V(F_1)$. It may be empty.
  2. $X_j^i = (\pi_{i,i-1}^{-1}(X_j^{i-1}) \cap V(F_i)) \cup \pi_{i,i-1}^{-1}(X_{j-1}^{i-1})$ for $1 \leq j < i$.
- Every connected component S of $X_j^i \setminus X_{j-1}^i$ is a locally closed j-dimensional $\mathbb{K}$-analytic submanifold of U_i and hence defines a stratification $\mathcal{S}_i$ of U_i. In particular we call $\mathcal{S} = \mathcal{S}_n$ *the canonical stratification associated to a system of pseudopolynomials*.

Example.

- Given a polynomial $F \in \mathbb{K}[x_1, ..., x_n]$ we construct an associated system of polynomials $F_i \in \mathbb{K}[x_1, \ldots, x_i]$, $i = 1, ..., n$, as follows.
  First we set $F_n = F$ that after a linear change of coordinates we may assume monic in x_n. Then let F_{n-1} be the discriminant of $F_{n,red}$.
- We again make a linear change of coordinates $x_1, ..., x_{n-1}$ so that we may assume F_{n-1} monic in x_{n-1} and we continue until we get F_j a non-zero constant.
- This construction is algorithmic except taking generic system of coordinates. For such a system of polynomials $F_i \in \mathbb{K}[x_1, \ldots, x_i]$, $i = 1, ..., n$, we may consider the canonical stratification described above.

Zariski equisingularity.

Theorem (Puiseux with parameter, Jung, Zariski)

Let

$$F(t,x,z) = z^p + \sum_{j=1}^{p} A_j(t,x) z^{p-j}$$

be complex analytic, where $x \in \mathbb{C}$, $z \in \mathbb{C}$, $A_j \in \mathbb{C}\{t,x\}$, *where* $t \in \mathbb{C}^l$ *parameter.*

Suppose that discriminant of F *is of the form* $\Delta_F(t,x) = x^M unit(t,x)$.
Then there is $d \in \mathbb{N}$ *and* $\tilde{a}_j(t,y) \in \mathbb{C}\{t,y\}$ *s.t.*

$$F(t,y^d,z) = \prod_{j=1}^{p} (z - \tilde{a}_j(t,y)).$$

Corollary (preservation of multiplicities)

For x_0 fixed, the roots of $F(t, x_0, z)$

$$t \to a_1(t, x_0), \ldots, a_p(t, x_0)$$

can be chosen continuous and of constant multiplicities.

Zariski equisingularity

Given a system of complex analytic germs

$$F_i(t, x^i) = x_i^{p_i} + \sum_{j=1}^{p_i} A_{i-1,j}(t, x^{i-1}) x_i^{p_i - j}, \qquad i = 0, ..., n,$$

$t \in \mathbb{C}^l$, $x^i = (x_1, ..., x_i) \in \mathbb{C}^i$, $A_{i-1,j} \in \mathbb{C}\{t, x^{i-1}\}$, $A_{i-1,j}(t, 0) \equiv 0$, s.t.

the family is **Zariski equisingular** along $T = \mathbb{C}^l \times \{0\}$ at the origin if

1. $F_{i-1}(t, x^{i-1}) = 0$ is the Weierstrass polynomial associated to the discriminant $\Delta(F_{i,red})$ of F_i reduced.
2. There is i, s.t. $F_i(0) \neq 0$.

Zariski equisingularity implies topological triviality

Theorem (Varchenko)

If the family $\{F_i(t, x^i)\}$ is Zariski equisingular then the family of set germs $t \to V_t(F_n) = \{F_n(t, x) = 0\}$ is ***ambient topologically trivial****.*

i. e. there exist neighbourhoods $0 \in B \subset \mathbb{C}^l$, $0 \in \Omega_0 \subset \mathbb{C}^n$ and $0 \in \Omega \subset \mathbb{C}^l \times \mathbb{C}^n$, and a homeomorphism

$$\Phi(t, x) : B \times \Omega_0 \to \Omega,$$

- $\Phi(t, 0) = (t, 0)$ and $\Phi(0, x) = (0, x)$
- $\Phi(t, x) = (t, \Psi_1(t, x^1), \Psi_2(t, x^2), \dots, \Psi_n(t, x^n))$
- preserving the zero set of F: $\Phi(T \times V_0(F_n)) = V(F_n)$.
- moreover, preserving the zero set of any factor of F_n.

Theorem (preservation of order and of multiplicities, Zariski)

Let $U \subset \mathbb{C}^n$ be open and let $F(x,z) \in \mathcal{O}_U[z]$ be a monic polynomial in z. Suppose the discriminant $\Delta_F(x) \not\equiv 0$.

If the order $\operatorname{ord}_x \Delta_F$ is constant for $x \in S$, $S \subset U$ connected, then the roots of F

$$S \ni x \to a_1(x), \ldots, a_p(x)$$

can be chosen continuous and of constant multiplicities as the roots of F.

Moreover, the orders of F, $\operatorname{ord}_{(x,a_i(x))} F$ are constant.

Whitney Interpolation.

Given $a = (a_1, \dots, a_p) \in \mathbb{C}^p$, $b = (b_1, \dots, b_p) \in \mathbb{C}^p$, such that if $a_i = a_j$ then $b_i = b_j$. Define $\psi_{a,b} : \mathbb{C} \to \mathbb{C}$

$$\psi_{a,b}(z) := z + \frac{\sum_{i=1}^{p} \mu_i(z)(b_i - a_i)}{\mu(z)}, \quad \psi(a_i) = b_i,$$

where $\mu_i(z) = |z - a_i|^{-1}$, $\mu(z) = \sum_{j=1}^{p} \mu_j(z)$.

Then $\psi_{a,b} : \mathbb{C} \to \mathbb{C}$ is a bi-lipschitz homeomorphism provided

$$\gamma = \max_{a_i \neq a_j} \frac{|(b_i - a_i) - (b_j - a_j)|}{|a_i - a_j|} < (2p)^{-1}.$$

Moreover, $\psi_{a,b}$ depends continuously on a and b.

Trivializing family of plane curves.

Recall that under the assumption of Puiseux with parameter the roots of $F(t,x,z)$

$$t \to a_1(t,x), \dots, a_p(t,x)$$

can be chosen continuous in t and of constant multiplicities. Thus

$$\psi_{a(0,x),a(t,x)}(z) : \mathbb{C} \to \mathbb{C}$$

is a continuous (in t, x) family of self-homeomorphisms of $\mathbb{C}$.

Then there exist neighbourhoods $0 \in B \subset \mathbb{C}^l$, $0 \in \Omega_0 \in \mathbb{C}^2$ and $0 \in \Omega \subset \mathbb{C} \times \mathbb{C}^2$ such that $\Phi(t,x,z) : B \times \Omega_0 \to \Omega$

$$\Phi(t,x,z) = (t, x, \psi_{a(0,x),a(t,x)}(z)).$$

satisfies

- Φ is a homeomorphism;
- Φ preserves the zero set of F: $\Phi(T \times V_0(F)) = V(F)$.

Statement of results (A. Parusiński,-)

Theorem

If $F(t,x)$ can be completed ($F_n = F$) to a Zariski equisingular family then there exist neighbourhoods $0 \in B \subset \mathbb{C}^l$, $0 \in \Omega_0 \subset \mathbb{C}^n$ and $0 \in \Omega \subset \mathbb{C}^l \times \mathbb{C}^n$, and a homeomorphism

$$\Phi(t,x) : B \times \Omega_0 \to \Omega,$$

- $\Phi(t,0) = (t,0), \Phi(0,x) = (0,x)$
- Φ *is complex analytic in* t.
- Φ *and* Φ^{-1} *are real arc-analytic:*
  - *if* $x(s) : (-1,1) \to \Omega_0$ *is analytic then* $\Phi(t,x(s)) : B \times (-1,1) \to \Omega$ *is analytic.*
  - $\forall (t(s),x(s)) : (-1,1) \to \Omega$ *analytic there is* $\tilde{x}(s) : (-1,1) \to \Omega_0$ *such that* $(t(s),x(s)) = \Phi(t(s),\tilde{x}(s))$.
- *There are constants* $C, c > 0$ *s.t.*

$$c|F(\Phi(0,x))| \leq |F(\Phi(t,x))| \leq C|F(\Phi(0,x))|$$

In particular Φ *preserves the zero set of* F.

Idea of proof.

- Replace in

$$\psi_{a(0,x),a(t,x)}(z) = z + \frac{\sum_{i=1}^{p} \mu_i(z)(a_i(0,x) - a_i(t,x))}{\sum_{i=1}^{p} \mu_i(z)}$$

  $\mu_i(z) = |z - a_i|^{-1}$ by different functions of $(z - a_i)^{-1}$, that are in particular real rational functions.

- Using induction on n lift a trivialization $\Phi'(t, x^{n-1}) : B \times \Omega'_0 \to \Omega'$, to

$$\Phi(t, x^n) : B \times \Omega_0 \to \Omega,$$

  first to $V(F_n)$ (preservation of multiplicities)

- and then to $B \times \Omega_0$ using Whitney Interpolation

$$\Phi(t, x^{n-1}, x_n) = (\Phi'(t, x^{n-1}), \psi_{a(0,x^{n-1}),a(\Phi'(t,x^{n-1}))}(x_n)).$$

- Show the required properties of Φ by Puiseux with parameter theorem (on any real analytic curve $x^{n-1}(s)$).

Whitney Fibering Conjecture.

In 1965 Whitney stated the following conjecture.

Conjecture

Any analytic subvariety $V \subset U$ (U open in $\mathbb{C}^n$) has a stratification such that each point $p_0 \in V$ has a neighbourhood U_0 with a semi-analytic fibration.

By a semi-analytic fibration Whitney meant the following:
Let p_0 belong to a stratum M and let $M_0 = M \cap U_0$. Let N be the analytic plane orthogonal to M at p_0 and let $N_0 = N \cap U_0$. Then Whitney requires that there exists a homeomorphism

$$\phi(p,q) : M_0 \times N_0 \to U_0,$$

complex analytic in p, such that $\phi(p, p_0) = p$ ($p \in M_0$) and $\phi(p_0, q) = q$ ($q \in N_0$), and preserving the strata. He also requires that for each $q \in N_0$ fixed, $\phi(\cdot, q) : M_0 \to U_0$ is a complex analytic embedding onto an analytic submanifold $L(q)$ called the fiber (or the leaf) at q, and thus U_0 fibers continuously into submanifolds complex analytically diffeomorphic to M_0.

Whitney Fibering Conjecture.

Whitney also wants that this fibration induces stratifying conditions (a) and (b). To have the condition (b), quoting Whitney, "one should probably require more than just the continuity of ϕ in the second variable".

Theorem (A. Parusiński,-)

Any analytic subvariety $V \subset U$ (U open in $\mathbb{C}^n$ or $\mathbb{R}^n$) admits locally a stratification satisfying Whitney fibering conjecture. For algebraic varieties such a stratification can be chosen global.

According to Bekka every c-regular stratification is locally topologically trivial along strata (and conical).

Thus, for a c-regular stratified sets (Z, Σ), Σ a stratification of a closed set $Z \subset M$, for every point x in a stratum X, there is a neighbourhood U of x in M, a stratified set L and a homeomorphism

$h : (U, U \cap Z, U \cap X) \to (U \cap X) \times (\mathbb{R}^k, cL, v)$, given by
$h(z) = (\pi_X(z), \rho_X(z), \theta(z))$ where cL is the cone on L with vertex v.

If we fix the values of ρ_X and θ, then $\{z | \rho_X(z) = \rho, \theta(z) = \theta\}$ is a leaf diffeomorphic to $U \cap X$.

Recently Murolo-du Plessis-Trotman proved that given a c-regular stratified set we can choose h such that the tangent spaces to the leaves vary continuously on U, in particular as points tend to X.

For fix θ, then $\{z | \theta(z) = \theta\}$ is a wing, a C^0 manifold with boundary $U \cap X$ and smooth interior.
Then one can choose h so that the tangent spaces to the wings vary continuously and each wing is itself c-regular.

For more details one can look at the latest upload of C. MUROLO, A. Du PLESSIS, D.J.A. TROTMAN, the smooth version of Whitney's fibering conjecture: https://hal.archives-ouvertes.fr/hal-01359601.

THE SMOOTH WHITNEY FIBERING CONJECTURE AND OPEN BOOKS IN WHITNEY AND BEKKA STRATIFICATIONS.

BON COURAGE DAVID !

HANG IN THERE D.J.A. !