On the multiplicities of families of non-isolated hypersurface singularities

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Singularity theory and regular stratifications
on the occasion of David Trotman's retirement

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f_t topologically V-constant means that the family of hypersurfaces $V(f_t) = f_t^{-1}(0)$ is topologically trivial.

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Setting the problem

Let $B \subset \mathbb{C}^n$ and $D \subset \mathbb{C}$ be open balls around the origin, $z := (z_1, \ldots, z_n)$ linear coordinates for $\mathbb{C}^n$ and

$$f : (B \times D, \{0\} \times D) \to (\mathbb{C}, 0), \ (z, t) \mapsto f_t(z) := f(z, t),$$

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- $m_0(f_t)$: multiplicity of $V(f_t) := f_t^{-1}(0)$ at $0 \in \mathbb{C}^n$
- $m_0(f_t) = ord(f_t)$ at 0, where $ord(f_t)$ is the lowest degree in the power series expansion of f_t at 0.

- f_t is *topologically V-constant* (or $V(f_t)$ is topologically trivial) if for all sufficienty small t, there are neighbourhoods $U_0,\ U_t \subset B$, around the origin, and homeomorphism $\phi_t : (U_t, 0) \to (U_0, 0)$ such that

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Conjecture 1: If the family f_t is topologically V-constant, then it is equimultiple.

Hypersurfaces with isolated singularity

If $V(f_t)$ is a family with isolated singularity at the origin, by Lê-Ramanujam Theorem, provided that $n \neq 3$, it follows that Zariski multiplicity conjecture is equivalent to the following conjecture by Teissier:

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Theorem A: (G-M. Greuel (1986); C. Plénat and D. Trotman (2013)

If $f(z,t) = f_0(z) + tg_1(z) + t^2 g_2(z) + \ldots + t^r g_r(z) + \ldots$ is an analytic one parameter family of isolated hypersurface singularities with constant Milnor number at $z = 0$, and $m_0(f_0) = m$, then

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Theorem B: (G-M. Greuel (1986); O'Shea (1987))

Let f_0 be a quasihomogeneous polynomial with isolated singularities and f_t a μ-constant deformation of f_0. Then $m_0(f_t) = m_0(f_0)$.

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Theorem C: (Plénat-Trotman(2013))

Let $f(z,t) = f_0(z) + tg(z) + t^2 h(z)$ be a μ-constant family. If the singular set of the tangent cone of $\{f_0 = 0\}$ is not contained in the tangent cone of $\{h = 0\}$, then the multiplicity $m_0(f_t)$ is constant.

Theorem: (Lê-Saito, Teissier)

$F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$, $F_0(z) = f(z)$, $\mu(F_t) < \infty$. The following statements are equivalent.

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(2) For every holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n \times \mathbb{C}, 0)$

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(3) $\frac{\partial F}{\partial t} \in \overline{\mathcal{J}_F}$, ($\mathcal{J}_F = \langle \frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_n} \rangle$ is the Jacobean ideal in $\mathcal{O}_{n+1}$).

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(4) The polar curve of F with respect to $\{t = 0\}$ does not split i.e.

$$\Gamma_f = \{(z, t) \in \mathbb{C}^n \times \mathbb{C} \,|\, \frac{\partial F}{\partial z_i}(z, t) = 0, i = 1 \ldots, n\} = \{0\} \times \mathbb{C} \; near \; (0, 0).$$

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$\dim \Sigma F_t = 1, \ \ \Sigma F_t = V(f, g_t).$

F_t topologically $\mathcal{R}$-trivial, $\lambda_z(F_t)$ not constant.

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Example

(Fernández-Bobadilla,Gaffney (2008), Fernández-Bobadilla (2013))
Let $f, g_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ defined by
$f(x, y, z) = x^{15} + y^{10} + z^6$, $g_t(x, yz) = xy + tz$,
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Theorem (Massey, Theorem 6.5, LNM 1615)

If the family f_t is λ_z-constant, then $\{0\} \times D$ satisfies Thom's a_f condition at the origin with respect to the ambient stratum, that is, if p_k is a sequence of points in $(B \times D) \setminus \Sigma f$, such that $p_k \longrightarrow (0,0)$ and $T_{p_k} V_{f-f(p_k)} \longrightarrow T$, then $0 \times D \subset T$.

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Lemma (Eyral and R. (2015)) - Thom's inequalities

If $\{0\} \times D$ satisfies Thom's a_f condition at the origin with respect to the ambient stratum, then, for any holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n \times \mathbb{C}, 0)$, not contained in Γ_f, we have

$$\mathrm{ord}(\frac{\partial f}{\partial t} \circ \gamma) > inf\{\mathrm{ord}(\frac{\partial f}{\partial z_i} \circ \gamma) \mid i = 1, \ldots, n\}.$$

First Problem

Theorem A_{ni}: Eyral and R. (2015)

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Let $f(z,t) = f_0(z) + tg_1(z) + t^2 g_2(z)$ be a λ_z-constant family. If the singular set of the tangent cone of $\{f_0 = 0\}$ is not contained in the tangent cone of $\{g_2 = 0\}$, then the multiplicity $m_0(f_t)$ is constant.

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$$\text{ord}(\frac{\partial f}{\partial t} \circ \gamma) = \text{ord}(\frac{\partial f}{\partial t}) = \inf(m_0(g_j) + j - 1)$$

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As $\text{in}(\frac{\partial f}{\partial z_{i_0}} \circ \gamma) \neq 0$, the set $\gamma(\mathbb{C})$ is not contained in Γ_f, and it follows that $\quad m - 1 < j + m_0(g_j) - 1 \quad$ for every $j \geq 1$.

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As λ_z is constant, for all sufficiently high integers $0 \ll N_1 \ll N_2 \ll \ldots \ll N_d,\ d = \dim \Sigma_{f_t}$, the functions

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Suppose that $f(z,t) = f_0(z) + tg_1(z) + t^2 g_2(z)$, *with* $g_2 \neq 0$ *and* $\Sigma in(f_0) \nsubseteq C(V(g_2))$ *then* $\Sigma in(f_0) \times \mathbb{C} \nsubseteq \Gamma_f$.

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Observe that $\dim \Sigma \text{in}(f_0) \geq 1$ and by Theorem A_{ni}, $m_0(g_1) \geq m$ and $m_0(g_2) \geq m - 1$.

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Suppose that $f(z,t) = f_0(z) + tg_1(z) + t^2 g_2(z)$, with $g_2 \neq 0$ and $\Sigma in(f_0) \not\subseteq C(V(g_2))$ then $\Sigma in(f_0) \times \mathbb{C} \not\subseteq \Gamma_f$.

Observe that $\dim \Sigma \text{in}(f_0) \geq 1$ and by Theorem A_{ni}, $m_0(g_1) \geq m$ and $m_0(g_2) \geq m - 1$.

Suppose (by contradiction) that $m_0(g_2) = m - 1$.
By the previous lemma, there exists an index i_0 such that the restriction of $\frac{\partial f}{\partial z_{i_0}}$ to $\Sigma \text{in}(f_0) \times \mathbb{C}$ is $\neq 0$.

So we can pick a point $(z_0, t_0) \neq (0,0)$ in $\Sigma \text{in}(f_0)$ such that for all $s \neq 0$ sufficiently small,

$$\text{in}\frac{\partial f}{\partial t}(sz_0, st_0) \neq 0, \quad \text{in} g_2(sz_0, st_0) \neq 0, \quad \frac{\partial f}{\partial z_{i_0}}(sz_0, st_0) \neq 0$$

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Let $\gamma(s) = (sz_0, st_0)$, then we can check that the a_f condition fails along γ.

Second problem

In this second part, I discuss the extension of Theorem B (Greuel, 1986), (O'Shea, 1987) to the non-isolated case.

Also assume that for any $t \neq 0$ the polar curve $\Gamma^1_{f_t,z}$ is irreducible. Under these assumptions, if furthermore the families f_t and $f_{V(z_1)}$ are both topologically equisingular, then they are both equimultiple.

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Theorem

Suppose that f_t is a family of line singularities such that f_0 is weighted homogeneous with respect to a system of positive integer weights $(w_1, \ldots, w_n)$ satisfying the following conditions:

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Surfaces in $\mathbb{C}^3$ with smooth normalisation

$$X = \Phi^{-1}(0) \subset (\mathbb{C}^3 \times \mathbb{C}, 0),$$

family of reduced hypersurfaces in $\mathbb{C}^3$, defined by $\Phi : (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$.

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Question: If X is topologically equisingular, does it follow that $m_0(X_t)$ is constant ?

For a family of surfaces $X \subset (\mathbb{C}^3 \times \mathbb{C}, 0)$ whose normalisation is smooth, we can associate a family of parametrizations $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ whose images are X_t. More precisely,

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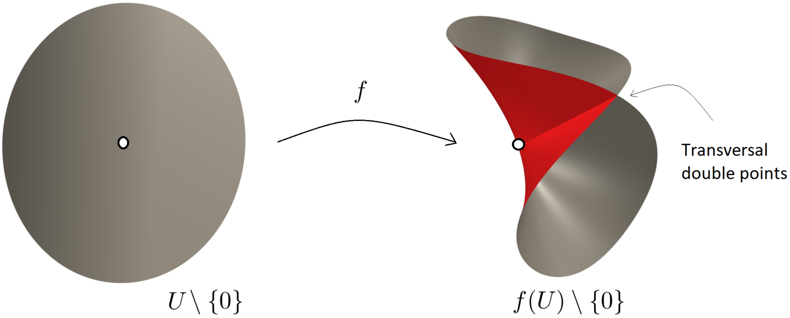
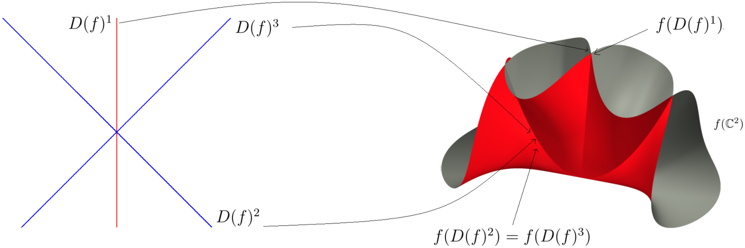
We consider 1-parameter unfoldings F of $\mathcal{A}$-finitely determined map-germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$.

(Mather-Gaffney geometric criterion) $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is $\mathcal{A}-$finitely determined if and only if for all representative of f, there exists a neighborhood U of 0 in $\mathbb{C}^2$ such that



the singularities of $f(U) \setminus \{0\}$ are just transversal double points.

Example: $f(x,y) = (x, y^2, xy^3 - x^3y)$, the singularity C_3 of Mond's list



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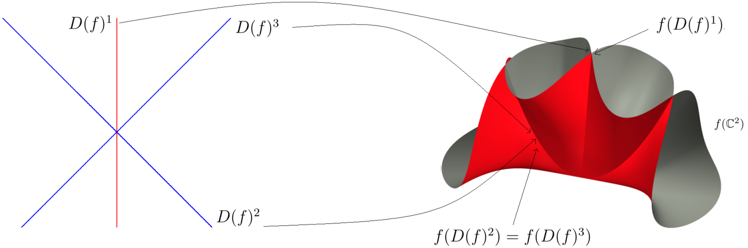

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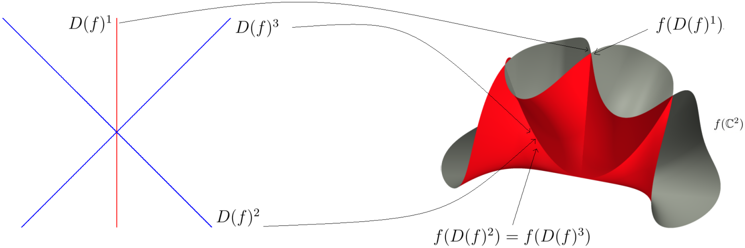

The double point curve is denoted by $D(f)$.

Double point set

$$D(f) := \left\{(x,y) \in U \,:\, f^{-1}(f(x,y)) \neq \{(x,y)\} \cup \Sigma(f)\right\}$$

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Definition:

F is a μ-constant unfolding if $\mu(D(f_t))$ is independent of t.

Topological triviality

We say F is a $\mathcal{A}$-topologically trivial if there are germs of homeomorphisms H and K such that

$$\begin{array}{ccc} (\mathbb{C}^2 \times \mathbb{C}, 0) & \xrightarrow{F} & (\mathbb{C}^3 \times \mathbb{C}, 0) \\ H \uparrow & \circlearrowleft & \downarrow K \\ (\mathbb{C}^2 \times \mathbb{C}, 0) & \xrightarrow{f \times Id} & (\mathbb{C}^3 \times \mathbb{C}, 0) \end{array}$$

where H and K are unfoldings of the identity.

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The strata in the source are the following:

$$\{\mathbb{C}^2\times\mathbb{C}\setminus D(F),\ D(F)\setminus T,\ T\}$$

In the target, the strata are

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Notice that F preserves the stratification, that is, F sends a stratum

Definition

An unfolding F as above is *Whitney equisingular* if the above stratifications in source and target are Whitney equisingular along T.

(b) F is topologically trivial

(c) F is Whitney equisingular.

Question: Does it follow that $(a) \iff (b) \iff (c)$? Answer: No.

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where $Y_0 := f(\mathbb{C}^2) \cap H$, and H is a generic plane in $\mathbb{C}^3$, passing through the origin.

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• Marar and Nuño-Ballesteros. A note on finite determinacy for corank 2 map germs from surfaces to 3-space, *Math. Proc. Cambr. Phil. Soc.*, (2008).

• Marar, Nuño-Ballesteros and Peñafort-Sanchis. Double point curves for corank 2 map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$. *Topology Appl.*, (2012).

• Marar and Nuño-Ballesteros. Slicing corank 1 map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$. *Quart. J. Math.*, (2014).

• Peñafort-Sanchis. Reflection Maps, *Mathematische Annalen*, (2020).

Among others...

O. N. Silva (Thesis), O. N. Silva and R. (2019)

$H = V(aX + bY + cZ)$ generic hyperplane,
$(Y_t, 0) = V(a(x^2 + txy) + b(x^2y + xy^2 + y^3) + c(x^5 + y^5))$

- $\mu(Y_0, 0) = 2$ and $\mu(Y_t, 0) = 1$ for $t \neq 0$, so $\mu_1(f_t(\mathbb{C}^2))$ not constant.
- $m_0(f_t(D(f_t))) = 22$ and $m_0(f_t(\mathbb{C}^2)) = 6$ for all t.

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Example

$f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0),\ f(x, y) = (x^2,\ x^2y + xy^2 + y^3,\ x^5 + y^5)$, and

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- $\mu(D(f_t))=441$ for all t, then F is topologically trivial.

  $H=V(aX+bY+cZ)$ generic hyperplane,
  $(Y_t,0)=V(a(x^2+txy)+b(x^2y+xy^2+y^3)+c(x^5+y^5))$
- $\mu(Y_0,0)=2$ and $\mu(Y_t,0)=1$ for $t\neq 0$, so $\mu_1(f_t(\mathbb{C}^2))$ not constant.
- $m_0(f_t(D(f_t)))=22$ and $m_0(f_t(\mathbb{C}^2))=6$ for all t.

Hence F is not Whitney equisingular.

F_t	μ		m_0	
(Corank 1 case)	$(\tilde{Y}_0, 0)$	$(\tilde{Y}_t, 0)$	$f(D(f))$	$f_t(D(f_t))$
$(x, y^4, x^5y + xy^5 + y^6 + ty^7)$	0	0	9	8
$(x, y^6, x^{13}y + xy^{13} + y^{14} + ty^{15})$	0	0	35	33
(Corank 2 case)				
$(x^2 + txy, x^2y + xy^2 + y^3, x^5 + y^5)$	2	1	22	22
$(x^3, y^5, x^2 - xy + y^2 + tx^2)$	1	1	23	22

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- Let $F = (f_t, t)$ be an unfolding of f. Then

F is topologically trivial $\Leftrightarrow$ F is Whitney equisingular $\Leftrightarrow$ $\mu(D(f_t))$ is constant.

Congratulations, David!!