On Sobolev spaces of bounded subanalytic manifolds

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Singularity Theory and Regular Stratifications
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We will consider the following Sobolev space of bounded subanalytic manifold M:

$$\mathcal{W}^{1,p}(M) := \{u \in L^p(M),\ |\partial u| \in L^p(M)\},$$

where ∂u stands for the gradient of u in the sense of distributions.

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$$\mathcal{W}^{1,p}(M) := \{u \in L^p(M),\ |\partial u| \in L^p(M)\},$$

where ∂u stands for the gradient of u in the sense of distributions. It is well known that this space, equipped with the norm

$$||u||_{\mathcal{W}^{1,p}(M)} := ||u||_{L^p(M)} + ||\partial u||_{L^p(M)}$$

is a Banach space, in which $\mathscr{C}^\infty(M)$ is dense for all $p \in [1, \infty)$.

Definition

We say that M is **connected at** $x \in \delta M = \overline{M} \setminus M$ *if* $\mathrm{B}(x,\varepsilon) \cap M$ *is connected for all* $\varepsilon > 0$ *small enough.*

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We say that M is **normal** *if it is connected at each* $x \in \delta M$.

Theorem

If M is normal then for all $p \in [1, \infty)$ sufficiently large $\mathscr{C}^\infty(\overline{M})$ is dense in $\mathcal{W}^{1,p}(M)$.

Assume that M is normal and let A be a subanalytic subset of δM. For all $p \in [1, \infty)$ sufficiently large, we have:

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Theorem

The linear operator

$$\mathscr{C}^\infty(\overline{M}) \ni \varphi \mapsto \varphi_{|A} \in L^p(A, \mathcal{H}^k), \qquad k := \dim A,$$

is bounded for $|| \cdot ||_{\mathcal{W}^{1,p}(M)}$ *and thus extends to a mapping* $\mathrm{tr}_A : \mathcal{W}^{1,p}(M) \to L^p(A, \mathcal{H}^k)$.

Assume that M is normal and let A be a subanalytic subset of δM. For all $p \in [1, \infty)$ sufficiently large, we have:

Theorem

If $\mathcal{S}$ is a stratification of A, then $\mathscr{C}^{\infty}_{\overline{M}\setminus\overline{A}}(\overline{M})$ is a dense subspace of $\bigcap_{Y \in \mathcal{S}} \ker \operatorname{tr}_Y$.

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In particular, in the case where A is dense in δM, we get:

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In particular, in the case where A is dense in δM, we get:

Corollary

If Σ is a stratification of a dense subset of δM then $\mathscr{C}^\infty_0(M)$ is dense in $\bigcap_{S \in \Sigma} \ker \operatorname{tr}_S$ for all $p \in [1, \infty)$ sufficiently large.

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$\mathscr{C}^\infty(\overline{M})$ is dense in $\mathcal{W}^{1,p}(M)$ for arbitrarily large values of p if and only if M is normal.

If the manifold M is unbounded, the trace is well-defined and it is L^p_{loc} on the boundary.
Since we can use cutoff functions, the density results remain true in the unbounded case.

Definition

A $\mathscr{C}^\infty$ normalization of M *is a definable* $\mathscr{C}^\infty$ *diffeomorphism* $h : \check{M} \to M$ *satisfying* $\sup_{x\in\check{M}} |D_x h| < \infty$ *and* $\sup_{x\in M} |D_x h^{-1}| < \infty$, *with* $\check{M}$ *normal* $\mathscr{C}^\infty$ *submanifold of* $\mathbb{R}^k$, *for some* k.

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Proposition

Every bounded definable manifold admits a $\mathscr{C}^\infty$ *normalization.*

Fix a normalization $h : \check{M} \to M$. As h and h^{-1} have bounded derivative the mapping $h_* : \mathcal{W}^{1,p}(\check{M}) \to \mathcal{W}^{1,p}(M)$, $u \mapsto u \circ h^{-1}$ is a continuous isomorphism for all p. The Main Theorem thus immediately yields that for $p \in [1, \infty)$ sufficiently large, the space

$$\mathscr{C}^h(\overline{M}) := h_* \mathscr{C}^\infty(\overline{\check{M}}) = \{u \circ h^{-1} : u \in \mathscr{C}^\infty(\overline{\check{M}})\}$$

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is dense in $\mathcal{W}^{1,p}(M)$.
Although the functions of $\mathscr{C}^h(\overline{M})$ may fail to be smooth on $\overline{M}$, this ring is satisfying for many purposes. A given function v of this ring has the property that for every x_0 in $\overline{M}$ the restriction of v to a connected component U of $\mathrm{B}(x_0, \varepsilon) \cap M$, $\varepsilon > 0$ small, extends to a function which is Lipschitz with respect to the inner metric.

Proposition

There are stratifications $\check{\mathcal{S}}$ *and* $\mathcal{S}$ *of* $\delta\check{M}$ *and* δM *respectively such that for each* $S \in \mathcal{S}$, $\overline{h}^{-1}(S) = \bigcup_{i=1}^{j} S_i$, *where, for each* $i \leq j$, S_i *is a stratum of* $\check{\mathcal{S}}$ *on which* $\overline{h}$ *induces a diffeomorphism* $h_{S_i} : S_i \to S$ *satisfying* $\sup_{x \in S_i} |D_x h_{S_i}| < \infty$ *and* $\sup_{x \in S} |D_x h_{S_i}^{-1}| < \infty$.

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Let $l := \sup_{x \in M} \mathfrak{c}_M(x)$, fix $S \in \mathcal{S}$. We define $\mathrm{tr}_S : \mathcal{W}^{1,p}(M) \to L^p(S)^l$ by setting for $v \in \mathcal{W}^{1,p}(M)$ (and $p \in [1, \infty)$ large):

$$\mathrm{tr}_S v := \left((\mathrm{tr}_{S_1} v \circ h) \circ h_{S_1}^{-1}, \dots, (\mathrm{tr}_{S_j} v \circ h) \circ h_{S_j}^{-1}, 0, \dots, 0 \right).$$

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Theorem

Let $A \subset \delta M$ be a subanalytic set of dimension k. For $p \in [1, \infty)$ sufficiently large, the linear operator

$$\mathrm{tr}_A : \mathcal{W}^{1,p}(M) \to L^p(A, \mathcal{H}^k)^l,$$

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Proposition

Let $A \subset \delta M$ be subanalytic. If p is sufficiently large then, for every $v \in \mathcal{W}^{1,p}(M)$, the set of functions $\{\operatorname{tr}_{A,1} v, \ldots, \operatorname{tr}_{A,l} v\}$ does not depend on the chosen $\mathscr{C}^\infty$ normalization.

Theorem (G. Valette)

Let $X \subset \mathbb{R}^n$ be subanalytic and $x_0 \in X$. For $\varepsilon > 0$ small enough, there exists a Lipschitz subanalytic homeomorphism

$$H : x_0 * (\mathrm{S}(x_0, \varepsilon) \cap X) \to \overline{\mathrm{B}}(x_0, \varepsilon) \cap X,$$

satisfying $H_{|\mathrm{S}(x_0,\varepsilon)\cap X} = Id$, preserving the distance to x_0, and having the following metric properties:

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Theorem (G. Valette)

1. *The natural retraction by deformation onto* x_0

$$r : [0,1] \times \overline{\mathrm{B}}(x_0, \varepsilon) \cap X \to \overline{\mathrm{B}}(x_0, \varepsilon) \cap X,$$

defined by

$$r(s,x) := H(sH^{-1}(x) + (1-s)x_0),$$

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is Lipschitz. Indeed, there is a constant C such that for every fixed $s \in [0,1]$, the mapping r_s defined by $x \mapsto r_s(x) := r(s,x)$, is Cs-Lipschitz.

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Theorem (G. Valette)

2. *For each* $\delta > 0$*, the restriction of* H^{-1} *to* $\{x \in X : \delta \leq ||x - x_0|| \leq \varepsilon\}$ *is Lipschitz and, for each* $s \in (0, 1]$*, the map* $r_s^{-1} : \overline{B}(x_0, s\varepsilon) \cap X \to \overline{B}(x_0, \varepsilon) \cap X$ *is Lipschitz.*

Example

$$X := \{(x, y) \in [0, 1] \times \mathbb{R} : |y| \leq x^2\}$$

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with $x_0 = (0,0)$. For each $(x,y) \in x_0 * (S(0,1) \cap X)$, let

$$H(x,y) := (t(x,y)x,\ t^2(x,y)xy),$$

where

$$t(x,y) = \left(\frac{2x^2 + 2y^2}{x^2 + \sqrt{x^4 + 4x^2y^2(x^2+y^2)}} \right)^{1/2}.$$

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A straightforward computation yields that on $x_0 * (S(0,1) \cap X)$ we have $|\partial t(x,y)| \leq \frac{C}{x}$ for some positive constant C.

Łojasiewicz's inequality

Theorem

Let f and g be two globally subanalytic functions on a globally subanalytic set A with $\sup_{x\in A}|f(x)| < \infty$. Assume that $\lim_{t\to 0} f(\gamma(t)) = 0$ for every globally subanalytic arc $\gamma : (0,\varepsilon) \to A$ satisfying $\lim_{t\to 0} g(\gamma(t)) = 0$.
Then there exist $\nu \in \mathbb{N}$ and $C \in \mathbb{R}$ such that for any $x \in A$:

$$|f(x)|^\nu \leq C|g(x)|.$$

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1. For all $s \in (0,1)$ we have for almost all $x \in M^\varepsilon$:

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2. For each $v \in L^p(M^\varepsilon)$, $p \in [1,\infty)$, we have for all $\eta \in (0,\varepsilon]$ and all $s \in (0,1]$:

$$\frac{1}{C}\left(\int_0^\eta ||v||^p_{L^p(N^\zeta)} d\zeta\right)^{1/p} \leq ||v||_{L^p(M^\eta)} \leq C\left(\int_0^\eta ||v||^p_{L^p(N^\zeta)} d\zeta\right)^{1/p}.$$

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3. There exists $\nu \in \mathbb{N}$ such that for each $v \in L^p(M^\varepsilon)$, $p \in [1,\infty)$, $\eta \in (0,\varepsilon)$, and $s \in (0,1)$:
$$||v \circ r_s||_{L^p(N^\eta)} \leq Cs^{-\nu/p}||v||_{L^p(N^{s\eta})}.$$

The operator Θ^M

For $u \in \mathcal{W}^{1,p}(M^\varepsilon)$ and $x \in M^\varepsilon$ we set:

$$\Theta^M u(x) := \int_0^1 \frac{\partial(u \circ r)}{\partial s}(s,x)\, ds = \int_0^1 < \partial u(r_s(x)), \frac{\partial r}{\partial s}(s,x) > \, ds.$$

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Lemma

For p sufficiently large, the function $[0,1] \ni s \mapsto ||\frac{\partial(u \circ r_s)}{\partial s}||_{L^p(M^\varepsilon)}$ belongs to $L^1([0,1])$ for all $u \in \mathcal{W}^{1,p}(M^\varepsilon)$, so that $\Theta^M u$ is well-defined. Moreover, for $u \in \mathcal{W}^{1,p}(M^\varepsilon)$ and $\eta < \varepsilon$, we then have:

$$||\Theta^M u||_{L^p(N^\eta)} \lesssim \eta^{1-\frac{1}{p}} \, ||u||_{\mathcal{W}^{1,p}(M^\eta)}.$$

Proof of the lemma

For $u \in \mathcal{W}^{1,p}(M^\varepsilon)$, we have:

$$\int_0^1 ||\frac{\partial(u \circ r_s)}{\partial s}||_{L^p(M^\varepsilon)} ds = \int_0^1 \left(\int_{M^\varepsilon} \left| \frac{\partial(u \circ r)}{\partial s}(s,x) \right|^p dx \right)^{1/p} ds$$

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$$\lesssim \int_0^1 \left(\int_{M^\varepsilon} |\partial u(r_s(x))|^p \, dx \right)^{1/p} ds$$

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&\lesssim \int_0^1 \left(\int_{M^\varepsilon} |\partial u(r_s(x))|^p \, dx \right)^{1/p} ds \\
&\lesssim \int_0^1 s^{\frac{-\nu}{p}} ||\partial u||_{L^p(M^{s\varepsilon})} ds
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&\lesssim \int_0^1 \left(\int_{M^\varepsilon} |\partial u(r_s(x))|^p \, dx\right)^{1/p} ds \\
&\lesssim \int_0^1 s^{\frac{-\nu}{p}} ||\partial u||_{L^p(M^{s\varepsilon})} ds \\
&\leq ||\partial u||^p_{L^p(M^\varepsilon)} \int_0^1 |s|^{-\nu/p} ds,
\end{aligned}$$

which is finite for $p > \nu$.

Proof of the lemma

$$||\Theta^M u||_{L^p(N^\eta)} \leq \int_0^1 \left(\int_{N^\eta} |\partial u(r_s(x))|^p \left| \frac{\partial r}{\partial s}(s,x) \right|^p dx \right)^{1/p} ds$$

Proof of the lemma

$$\begin{aligned} ||\Theta^M u||_{L^p(N^\eta)} &\leq \int_0^1 \left(\int_{N^\eta} |\partial u(r_s(x))|^p \left| \frac{\partial r}{\partial s}(s,x) \right|^p dx \right)^{1/p} ds \\ &\overset{(1)}{\lesssim} \eta \int_0^1 \left(\int_{N^\eta} |\partial u(r_s(x))|^p \, dx \right)^{1/p} ds \end{aligned}$$

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||\Theta^M u||_{L^p(N^\eta)} &\leq \int_0^1 \left(\int_{N^\eta} |\partial u(r_s(x))|^p \left| \frac{\partial r}{\partial s}(s,x) \right|^p dx \right)^{1/p} ds \\
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&\overset{(3)}{\lesssim} \eta \int_0^1 |s|^{-\nu/p} ||\partial u||_{L^p(N^{s\eta})} ds \\
&\leq \eta \left(\int_0^1 |s|^{-\nu p'/p} ds \right)^{1/p'} \left(\int_0^1 ||\partial u||^p_{L^p(N^{s\eta})} ds \right)^{1/p}
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&\leq \eta \left(\int_0^1 |s|^{-\nu p'/p} ds \right)^{1/p'} \left(\int_0^1 ||\partial u||^p_{L^p(N^{s\eta})} ds \right)^{1/p} \\
&\lesssim \eta \left(\int_0^1 ||\partial u||^p_{L^p(N^{s\eta})} ds \right)^{1/p} \quad (\text{for} \quad p > \nu p')
\end{aligned}
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Proof of the lemma

$$\begin{aligned} ||\Theta^M u||_{L^p(N^\eta)} &\leq \dots \\ &\lesssim \eta \left(\int_0^1 ||\partial u||^p_{L^p(N^{s\eta})} ds \right)^{1/p} \quad (\text{for} \quad p > \nu p') \\ &= \eta^{1-1/p} \left(\int_0^\eta ||\partial u||^p_{L^p(N^t)} dt \right)^{1/p} \quad (\text{setting } t := s\eta) \end{aligned}$$

Proof of the lemma

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&= \eta^{1-1/p} \left(\int_0^\eta ||\partial u||^p_{L^p(N^t)} dt \right)^{1/p} \quad (\text{setting } t := s\eta) \\
&\overset{(2)}{\lesssim} \eta^{1-1/p} ||\partial u||_{L^p(M^\eta)}.
\end{aligned}$$

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Lemma

For p sufficiently large, $\mathscr{R}^M u$ is constant on every connected component of M^ε, for all $u \in \mathcal{W}^{1,p}(M^\varepsilon)$. Moreover, Θ^M and $\mathscr{R}^M$ are then continuous projections and, if u extends to a continuous function on $\overline{M^\varepsilon}$ then $\mathscr{R}^M u \equiv u(0)$.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open subanalytic subset. For each $p \geq 1$, there exists $C > 0$ such that for any $u \in W^{1,p}(\Omega)$ the following inequality holds

$$||u - u_\Omega||_p \leq C||\nabla u||_p,$$

where $u_\Omega := \frac{1}{|\Omega|} \int_\Omega u(x)\, dx$.

Lemma

Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected subanalytic subset. There exists a subanalytic map

$$h : \Omega \times [0,1] \to \Omega, \; (x,s) \mapsto h_s(x)$$

continuous with respect to the second variable and such that

1. $h_1(\Omega) \subset B(z,\alpha) \subset \Omega$ *for some* $\alpha > 0$ *and* $z \in \Omega$;
2. $d_x h_t$ *is invertible for almost every* $(x,t) \in \Omega \times [0,1]$, *and moreover there exists* $C > 0$ *such that whenever* $d_x h_t$ *is invertible, we have* $||d_x h_t^{-1}|| \le C$.

Lemma

Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected subanalytic subset. There exists a subanalytic family of continuous arcs $\gamma_{x,y} : [0,1] \to \Omega$, $x, y \in \Omega$, such that $\gamma(0) = x$, $\gamma(1) = y$ for each such x and y, and $||\gamma'_{x,y}(s)|| \leq C$ for all $s \in (0,1)$, and some constant C independent of x and y. Moreover, there is $\eta > 0$ such that for almost every $x, y \in \Omega$:

$$\begin{cases} jac(\Gamma_{s,y})(x) \geq \eta, & s \geq \frac{1}{2} \\ jac(\tilde{\Gamma}_{s,x})(y) \geq \eta, & s < \frac{1}{2} \end{cases}$$

where $\Gamma_{s,y} : \Omega \ni x \mapsto \gamma_{x,y}(s) \in \Omega$ and $\tilde{\Gamma}_{s,x} : \Omega \ni y \mapsto \gamma_{x,y}(s) \in \Omega$.

Theorem

For every definable family $\Omega \subset \mathbb{R}^n \times \mathbb{R}^k$ there is a constant C such that for all $t \in \mathbb{R}^k$ and all $u \in W^{1,p}(\Omega_t, \delta\Omega_t)$ we have:

$$||u||_{L^p(\Omega_t)} \leq C|\Omega_t|^{1/n}||\nabla u||_{L^p(\Omega_t)}.$$

Thank you !