University of Aix-Marseille
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Master's Thesis

ANALYSIS ON FOCK SPACES

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Summary: This Master's thesis aims to gather several parts and results of the graduate book written by Kehe Zhu on the analysis on Fock spaces ([31]). We mainly focus on the Toeplitz operators, the small Hankel operators and the (big) Hankel operators. Some preliminary results on entire functions, Fock spaces, Schatten class and Berezin transform are given at the beginning to introduce the terms used in the rest of this paper and to make it easier to understand.

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I. Introduction

The presentation of Fock spaces and the main results gathered in this paper closely follow Kehe Zhu's book (see [31]).

Fock Spaces are functional spaces, subspaces of the larger L^p space of analytic functions. Fock spaces are different from more standard Hardy and Bergman spaces in a sense that it's defined on the complex plane with the Euclidean metric and the reproducing kernel of the Fock L^2 space is of exponential form. It worth mentioning the role of Fock spaces in quantum mechanics to construct the quantum states of a variable or unknown number of identical particles even if this is not treated in this paper which focuses only on the mathematical aspects of Fock spaces.

We can mention several unique characteristics of Fock spaces like the unboundedness of the Fock kernel $e^{\alpha z\bar{w}}$, even when one of the two variables is fixed. It makes many estimates much more difficult but in the same time the exponential decay makes the convergence of certain integrals or infinite series much easier. We also face the absence of bounded analytic or harmonic functions other than the constant ones. In Fock spaces, in the theory of Hankel and Toeplitz operators, there is on "cutoff" point when characterizing membership in the Schatten classes. Finally, as analysis on such spaces takes place overall the complex plane, certain techniques and methods from Fourier analysis become available, like the relationship between Toeplitz operators and pseudodifferential operators on $L^2(\mathbb{R})$.

This paper starts with a chapter on a general introduction of the Fock spaces and some important results used in the rest of the article. We can represent Fock spaces as subspaces of $L^p(\mathbb{C}, d\lambda_\alpha)$ consisting of entire function or as entire functions f such that $f(z)e^{-\alpha|z|^2/2}$ belongs to $L^p(\mathbb{C}, dA)$, which is equivalent. We introduce the orthonormal basis, the reproducing kernel and the normalized reproducing kernel. The boundedness of the integral operators Q_α and P_α have been studied in [17], [23], [27] and [28]. We also give some results on the atomic decomposition in F^p_α, stating that every function can be decomposed into an infinite series of kernel function, which has been studied in [23], and a maximum principle result whose proof can be found in [26].

This chapter has a second part focusing on the Berezin transform for operators and functions that has been introduced in [5] and studied in [22-26]. This tool is widely used to study Hankel operators, Toeplitz operators and composition operators, see [33]. In Fock spaces the Berezin transform, with good parameters, is the Heat transform. The semigroup property of the Berezin transform was observed in [12]. The Lipschitz property for bounded function was proven in [11] and for bounded linear operators in [14] and [15]. This boundedness with the semigroup property shows that Berezin transform on bounded function is a rapidly smoothing operation. However, we can find unbounded function fixed by the Berezin transform that are not harmonic. We also introduce the notion of Schatten classes which will be used in the next sections.

The second part also focuses on the Fock-Carleson measures, that have been first introduced in the Hardy space setting ([18]) and developed in [22], and on the notions of BMO and VMO. These notions using a fixed Euclidean radius were introduced in [13, 34]. The extension to the Fock space setting has been done in several steps starting from bounded symmetric domains [4], then strongly pseudo-convex domains in [24] then Bergman spaces on the unit ball in [32]. The Lipschitz estimate for the Berezin transform of a function in BMO was proved in [2].

The following chapter of this paper introduces the Toeplitz operators whose study started in [27, 28]. Note that in this chapter we only put ourselves in the Fock space F_α^2 setting. Preliminary results and Trace results come from [12]. It's also in this paper that the use of operator $T^{(t)}$, introduced in III.3, to study trace-class properties of Toeplitz operators, appeared first. The Bargmann transform, introduced and developed in the second part of this chapter, comes from [19]. Some boundedness and compactness result from sections III.3 and III.4 have been proven in [12] as well. The case of bounded and compact Toeplitz operators with nonnegative symbols has been studied in [22]. The compactness of Toeplitz operators with bounded symbol and their extension to BMO symbols, given in III.4, have been obtained respectively in [30] and [35].

In section III.5 we determine when the Toeplitz operator belong to the Schatten class S_p. We can split this section in two case: when $1 \leq p < \infty$ and when $0 < p < 1$. The first case is easy while the second case is more complex due to the different behavior in the Fock space setting compared to the Bergman space setting. More precisely the cutoff point when Schatten class Toeplitz operator are characterized with the Berezin transform disappears in the Fock space setting. The last part, section III.6, follows [25] which gives the characterization of finite-rank Toeplitz operators induced by compactly supported measures.

Chapter IV studies small Hankel operators h_φ on the Fock space F_α^2. We present results for boundedness, compactness and characterization of h_φ induced by entire functions belonging to the Schatten classes S_p. Like the previous chapter two main cases occur: when $1 \leq p < \infty$ and when $0 < p < 1$. The first case has been developed in [23] while the second case can be found in [29]. We also characterize h_φ whose rank is finite in part IV.4

The last main chapter studies big Hankel operators H_φ on the Fock space F_α^2. In a similar fashion we consider the problem of boundedness, compactness and membership in the Schatten classes S_p. The study of compactness for Hankel operators induced by bounded symbol on Fock spaces can be found in [10]. [34] introduced the study of BMO and VMO, defined with a fixed radius, into the study of Hankel and Toeplitz operators. The extension on Fock spaces has been done in [13].

One of the main results of the Fock space, proven in [10, 11] and not true in Bergman and Hardy spaces, states that when f is bounded, the Hankel operator H_f is compact on F_α^2 if and only if $H_{\bar{f}}$ is compact. The last part of this chapter, whose proofs are given in [21] and [30], presents when H_f and $H_{\bar{f}}$ belong to the Schatten class S_p. We will see that, because of the exponential decay of the Fock kernel, the cutoff point disappears in the Fock spaces setting contrary to the Bergman space case.

A conclusion of the main results is given in the last chapter of this paper.