MEMOIRE

MASTER 2

Singularity Theory

Whitney Stratification

of sets definable in the structure $\mathbb{R}_{exp}$

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THE MAIN POINTS DISCUSSED IN THIS TEXT

1. A brief Introduction

2. Preliminaries on o-minimal Structures

   This part consists of definitions, theorems and their corollaries indispensable for the purpose of the study of Whitney stratification.

3. Under the title of Whitney stratification, a significant section has been devoted to the Grassmannian of k-dimensional vector spaces of $\mathbb{R}^n$. The properties reviewed from Linear Algebra show how the Grassmannian are constructed and are used as algebraic subsets of $\mathbb{R}^{n^2}$. They serve as the main tools in the definition of Whitney conditions a) and b).

4. Whitney Conditions : their definitions.

   Then the theorem 2.5 . about the strict inequality of dimensions:

   $\dim(Y \setminus W_a(X,Y)) < \dim Y$ and $\dim(Y \setminus W_b(X,Y))$.

   The proof of that theorem is lenghy : it demands the proof of several lemmas and its use will be seen in the last theorem of the section :

   Theorem (Whitney Stratification).

5. A few examples complete the text: verification des conditions de regularité aux points singuliers.

INTRODUCTION

- Stratification theory is based on the 'natural' idea of dividing a singular space into manifolds. It deals with the study of topological spaces endowed with a partition by smooth manifolds satisfying specific conditions. Most of singular spaces appearing in Analysis have the structure of stratified spaces satisfying Whitney conditions.

- Whitney stratifications exist in different contexts: algebraic sets, semi-algebraic sets, analytic sets, and several others.

  Here the work will be done with a larger class of sets, namely definable sets, more precisely with sets definable in the language of rings, in short, in o-minimal structures.

- A Whitney stratification for $X \subset \mathbb{R}^n$ means a partition of $\mathbb{R}^n$ into a finite submanifolds $S_0, S_1, \ldots, S_n$ where S_d has dimension d, such that X is a union of some of the connected components of these sets, and such that certain regularity conditions called Whitney conditions related to different sets S_d are satisfied. Those are conditions we need to study in these pages and give some examples.

I.00 Preliminaries

O-minimal Structures

This is a brief summary of the theory of o-minimal structures, introducing the properties that will be used in later sections. Roughly, an o-minimal structure is a collection of 'tame' subsets of Euclidean space with which one can perform standarol geometric and topological constructions.

I.0 Definitions and elementary properties.

. Let A_n be the smallest ring of real-valued functions on $\mathbb{R}^n$ such that:

(a) A_n contains all polynomials, ie $\mathbb{R}[x_1, \dots, x_n] \subset A_n$

(b) A_n is closed under taking exponentiation, i.e. if $f \in A_n$, then $\exp f \in A_n$

I.1 Definition. A structure on $\mathbb{R}^n$ consists of a collection $\mathcal{D} = (\mathcal{D}_n)_{n \in \mathbb{N}}$ of subsets $\mathcal{D}_n$ of $\mathbb{R}^n$ for each $n \in \mathbb{N}$. $\mathcal{D}$ is the smallest class of subsets of Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$ such that:

(1) $\mathcal{D}_n$ is a Boolean algebra containing $\mathbb{R}^n$, ie, $\mathcal{D}$ is closed under intersection, union and complement.

(2) $\mathcal{D}_n$ contains all sets of the form $\{x \in \mathbb{R}^n : f(x) = 0\}$ where $f \in A_n$.

(3) $\mathcal{D}_n$ contains the diagonals $\{(x_1, \dots, x_n) \mid x_i = x_j\}$ for any $1 \le i < j \le n$

(4) if $X \in \mathcal{D}_n$ then $X \times \mathbb{R} \in \mathcal{D}_{n+1}$ et $\mathbb{R} \times X \in \mathcal{D}_{n+1}$.

(5) if $S \in \mathcal{D}_{n+1}$, then $\pi(S) \in \mathcal{D}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is projection onto the first n coordinates;