Institut Camille Jordan
Université Claude Bernard Lyon 1
Master 2 Internship 2020-2021

Two-dimensional entropy-preserving schemes for hyperbolic conservation laws

This Master internship is dedicated to the numerical approximation of weak solutions of hyperbolic systems of conservation laws:

$$\partial_t w + \nabla f(w) = 0, \tag{1}$$

where $w \in \Omega \subset \mathbb{R}^N$ is the state vector, function of the time and space variables $\mathbf{x} \in \mathbb{R}^d$, $t > 0$ and $f : \mathbb{R}^N \to \mathbb{R}^N$ is the flux function. It is well known that, due to the non linearity of the flux function, discontinuitites may develop in finite time and the uniqueness of a smooth solution is lost. In order to rule out nonphysical discontinuities, the system is classically endowed with entropy inequalities:

$$\partial_t \eta(w) + \nabla G(w) \leq 0, \tag{2}$$

where $\eta : \Omega \to \mathbb{R}$ is a convex function and $G : \Omega \to \mathbb{R}$ is the associated entropy flux.

Naturally, these issues reverberate from a numerical point of view, and one of the main difficulties when considering numerical approximations of (1) is to provide a discrete counterpart of (2). Even for first order accurate schemes, obtaining rigorous discrete entropy estimates is a difficult task and, still today, few schemes are able to meet this constraint; the property of entropy stability is generally not reachable (or even violated in some cases). Very recently, some advances have been realized in this direction in the one-dimensional case ($d = 1$), based on the concept of artificial viscosity ([Tadmor (1984), Tadmor (2016)]). The proposed method allows to recover the required entropy inequalities on the basis of any numerical solver (possibly entropy-violating), without any technical difficulty. First results were illustrated on the Euler equations [Berthon *et al*(2020)] and on the Shallow Water equations with source terms [Berthon *et al*(2019)]. Work is currently in progress for applications to non-conservative equations (particularly, for models of multiphase flows).

The current progress of these works motivates a two-dimensional extension, which is the main stake of the proposed internship. Based on the works previously mentioned, we aim the conception, analysis and implementation of two-dimensional entropy-satisfying schemes for the approximation of hyperbolic conservation laws. The properties of the obtained schemes will be illustrated through classical models such as the Euler equations or the Shallow Water equations. If structured meshes can be considered as a first step, the use of unstructured meshes appears quite appropriate in this context, since the works [Berthon (2006)] guarantee the possibility of extending stability results thanks to a judicious use of convexity inequalities.

As a prerequisite, the candidate should be familiar with scalar conservation laws and their approximation with finite volume numerical schemes. Some knowledge of the theory of hyperbolic systems of conservation laws is also valuable.

References

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Keywords: Fluid Mechanics, Hyperbolic Conservation Laws, Entropy Stability, Numerical Schemes, Unstructured Meshes.

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