POISSON BOUNDARY OF DISCRETE GROUPS

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Contents

0. Introduction

0.1. The classical *Poisson integral representation formula* for harmonic functions on the open unit disk $\mathbb{D}$ of the complex plane has the form

$$\varphi(z) = \int_0^1 \frac{1-|z|^2}{|e^{2\pi i\theta} - z|^2} F(\theta)\, d\theta = \int_0^1 \Pi(z,\theta) F(\theta)\, d\theta = \langle F, \nu_z \rangle \,, \tag{0.1}$$

where $d\nu_z(\theta) = \Pi(z,\theta)d\theta$ are the *harmonic measures* on $\partial\mathbb{D}$ associated with points $z \in \mathbb{D}$, and $\Pi(z,\theta)$ is the *Poisson kernel*. It recovers values of a continuous harmonic function $\varphi \in C(\overline{\mathbb{D}})$ from its boundary values $F \in C(\partial\mathbb{D})$. However, the right-hand side of (0.1) makes sense for any bounded measurable function $F \in L^\infty(\partial\mathbb{D})$, and the Poisson formula also establishes an isometry between the Banach space $H^\infty(\mathbb{D})$ of *all* bounded harmonic functions on $\mathbb{D}$ and the space $L^\infty(\partial\mathbb{D})$.

Since $\nu_{g(0)} = g\nu$ for any conformal automorphism g of $\mathbb{D}$, where $d\nu(\theta) = d\theta$ is the normalized Lebesgue measure on $\partial\mathbb{D}$, the Poisson formula can be rewritten as

$$\varphi(go) = \langle F, g\nu \rangle \,, \qquad g \in G \,, \tag{0.2}$$

where $G \cong SL(2,\mathbb{R})$ is the group of all conformal automorphisms of $\mathbb{D}$. Considering $\mathbb{D}$ as the Poincaré model of the hyperbolic plane $\mathbb{H}^2$ with the reference point $o \cong 0$ and

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the absolute $\partial\mathbb{H}^2 \cong \partial\mathbb{D}$, the Poisson formula becomes an isometry between the space $H^\infty(\mathbb{H}^2)$ of bounded harmonic functions on $\mathbb{H}^2$ and the space $L^\infty(\partial\mathbb{H}^2,\nu)$. The measure ν is the unique K-invariant measure on $\partial\mathbb{H}^2$, where $K = \operatorname{Stab} o \cong SO(2)$.

0.2. The space $H^\infty(\mathbb{H}^2)$ also admits a description in terms of a *mean value property*. Namely, a function φ belongs to $H^\infty(\mathbb{H}^2)$ iff

$$\varphi(x) = \int \varphi(y)\, d\pi_x(y) \qquad \forall\, x \in \mathbb{H}^2 \ , \tag{0.3}$$

where π_x is the uniform probability measure on the radius 1 circle in $\mathbb{H}^2$ centered at x. Denote by μ the bi-K-invariant probability measure on G such that $\mu = m_K * \delta_g * m_K$ for any $g \in G$ with $\operatorname{dist}(o, go) = 1$, where m_K is the Haar measure on K. Then a function φ on $\mathbb{H}^2$ satisfies (0.3) iff its lift to G defined as $f(g) \equiv \varphi(go)$ has the property that

$$f(g) = \int f(gh)\, d\mu(h) \qquad \forall\, g \in G \ , \tag{0.4}$$

Conversely, any function f on G satisfying (0.4) is right K-invariant, so that it is a lift of a function φ on $\mathbb{H}^2 \cong SL(2,\mathbb{R})/SO(2)$ which satisfies (0.3). Thus, formula (0.2) takes the form

$$f(g) = \langle F, g\nu \rangle \ , \qquad g \in G \ , \tag{0.5}$$

of an isometry between the space $H^\infty(G,\mu)$ of bounded *μ-harmonic functions*, i.e., of those that satisfy the mean value property (0.4), and the space $L^\infty(\partial\mathbb{H}^2,\nu)$. Since all integrals (0.5) are μ-harmonic functions, the measure ν satisfies the relation $\nu = \mu * \nu$ (such measures are called *μ-stationary*).

0.3. Given an arbitrary locally compact group G with a probability measure μ one can now ask whether there exists a G-space B with a probability measure ν on it such that formula (0.5) (which we still be calling the *Poisson formula*) establishes an isometric isomorphism between the space $H^\infty(G,\mu)$ of bounded μ-harmonic functions and the space $L^\infty(B,\nu)$. Under natural non-degeneracy and absolute continuity conditions such a space, indeed, exists and is unique. Below we shall refer to it as the *Poisson boundary* of the pair (G,μ) and denote it (Γ,ν).

The notion of the Poisson boundary was first introduced by Furstenberg [Fu63a], [Fu71] although in the context of general Markov chains (not necessarily group invariant) it can be traced back to earlier papers of Blackwell [Bl55] and Feller [Fe56]. The simplest way to define the Poisson boundary consists in putting this problem into a more general setup of finding integral representations for bounded invariant functions of Markov operators, see [Dy82], [Re84], [Ka92].

A function f is μ-harmonic if it is an invariant function of the Markov operator $P_\mu f(g) = \int f(gh)\, d\mu(h)$. The associated Markov chain on G (the right *random walk* determined by the measure μ) has transition probabilities $\pi_g = g\mu$, i.e., at each step the Markov particle jumps from a point $g \in G$ to the point gh, where h is a μ-distributed random *increment*. Thus, given the position x_0 of the random walk at time 0, its position x_n at time n is obtained by multiplying x_0 by independent μ-distributed increments h_i:

$$x_n = x_0 h_1 h_2 \cdots h_n \ . \tag{0.6}$$

Fix a reference probability measure θ on G equivalent to the Haar measure, and let $\mathbf{P}_\theta$ be the measure in the path space $G^{\mathbb{Z}_+}$ determined by the initial distribution θ, i.e., the image of the product measure $\theta \otimes \bigotimes_{i=1}^{\infty} \mu$ under the map (0.6). Then the one-dimensional distribution of the measure $\mathbf{P}_\theta$ at time n, i.e., the distribution of x_n, is the convolution $\theta\mu_n$, where μ_n is the n-th convolution power of μ. The measure $\mathbf{P}_\theta$ decomposes as an integral $\int \mathbf{P}_g \, d\theta(g)$ of measures $\mathbf{P}_g$ with starting points $g \in G$. By $\mathbf{E}_\theta, \mathbf{E}_g$ denote the expectations (the integrals) with respect to the measures $\mathbf{P}_\theta, \mathbf{P}_g$.

A function on G is μ-harmonic precisely if the sequence of its values along sample paths of the random walk is a *martingale* with respect to the increasing filtration of coordinate σ-algebras in the path space. Then, by the Martingale Convergence Theorem, for any $f \in H^\infty(G, \mu)$ and $\mathbf{P}_\theta$-a.e. sample path $\boldsymbol{x} = \{x_n\}$ there exists a limit $\widehat{F}(\boldsymbol{x}) = \lim f(x_n)$, which is invariant with respect to the time shift T in the path space. Conversely, for any T-invariant function $\widehat{F} \in L^\infty(G^{\mathbb{Z}_+}, \mathbf{P}_\theta)$ the conditional expectations

$$f(g) = \mathbf{E}_\theta(\widehat{F} | x_o = g) = \mathbf{E}_g \widehat{F} \tag{0.7}$$

yield a μ-harmonic function f such that a.e. $f(x_n) \to \widehat{F}(\boldsymbol{x})$, and we have an isometry between the space $H^\infty(G, \mu)$ and the subspace of T-invariant functions in $L^\infty(G^{\mathbb{Z}_+}, \mathbf{P}_\theta)$.

In the present paper we define the Poisson boundary Γ in a purely measure theoretical way as the *space of ergodic components of the time shift in the path space* by using the fact that the path space $(G^{\mathbb{Z}_+}, \mathbf{P}_\theta)$ is a *Lebesgue space* and the fundamental theorem of Rokhlin on correspondence between sub-σ-algebras, measurable partitions and quotient spaces for Lebesgue spaces, e.g., see [CFS82]. Let $\mathbf{bnd} : G^{\mathbb{Z}_+} \to \Gamma$ be the corresponding quotient map. We say that the measures $\nu_g = \mathbf{bnd}\, \mathbf{P}_g, \ g \in G$ are the *harmonic measures* on Γ. Then formula (0.7) takes the form $f(g) = \langle F, \nu_g \rangle$ of an isometry between the spaces $H^\infty(G, \mu)$ and $L^\infty(\Gamma, \nu_\theta)$, where $\nu_\theta = \mathbf{bnd}\, \mathbf{P}_\theta$, and $F(\mathbf{bnd}\, \boldsymbol{x}) = \widehat{F}(x)$.

The path space $G^{\mathbb{Z}_+}$ is provided with a coordinate-wise action of G commuting with the time shift T, so that the Poisson boundary comes endowed with a group action, and the boundary map $\mathbf{bnd}$ is equivariant. Let $\mathbf{P} = \mathbf{P}_e$ with e being the identity of G. Then $\nu_g = g\nu$, where $\nu = \mathbf{bnd}\, \mathbf{P}$, and finally we arrive precisely at the sought for Poisson formula (0.5).

The reference measure ν_θ is quasi-invariant with respect to the action of G, but the measure ν *a priory* does not have to be quasi-invariant or even absolutely continuous with respect to ν_θ (the integrals (0.5) are given sense using the notion of conditional decomposition of measures in Lebesgue spaces [Ka92]). If the measure μ is *spread out*, i.e., there exists a convolution power of μ non-singular with respect to the Haar measure, then ν is absolutely continuous with respect to ν_θ, but still need not be equivalent to ν_θ (if the closed semigroup generated by the support of μ is smaller than G). However, in the spread out case the measure ν_θ can be easily recovered from ν. Below we shall always mean by the Poisson boundary the measure space (Γ, ν), and do not require the measure ν to be quasi-invariant.

This construction is completely general and is applicable to any Markov operator on a Lebesgue space [Ka92]. It significantly clarifies the definition of the Poisson boundary and allows one to avoid a number of unnecessary complications (cf. [Az70], [Fu71]). Equivalent definitions of the Poisson boundary for random walks on groups can be given in terms of the Mackey range over the Bernoulli shift in the space of increments [Zi78], in terms of ideals in the group algebra of G [Wi90], or in terms of topological dynamics [DE90].

0.4. Having defined an abstract Poisson boundary, the next problem is to identify it with a certain concrete measure space associated with the group G and the measure μ.

For example, let $D \subset \mathbb{R}^d$ be a domain in a Euclidean space with boundary ∂D, and λ_x, $x \in D$ – the family of harmonic measures on ∂D (here the term "harmonic measure" is used in the classical sense). Then the map $f(x) = \langle F, \lambda_x \rangle$ determines an embedding of the space of bounded measurable functions on ∂D (with respect to the harmonic measure type) into the space $H^\infty(D)$ of bounded harmonic functions on D. When is this embedding an isomorphism, i.e., when can the Poisson boundary of D be identified with the geometric boundary ∂D? This question is well known in classical analysis, and it is already non-trivial in the case of the disk in $\mathbb{R}^2$, where the answer (yes) can be obtained by using an explicit form of the Poisson kernel [Ru80, Theorem 4.3.3]. For general Euclidean domains the problem was solved in [Bi91], [MP91].

Returning to the random walks, assume for a moment that the group G is equivariantly embedded into a topological space B, and $\mathbf{P}$-a.e. sample path $\boldsymbol{x} = \{x_n\}$ converges to a limit $x_\infty = \pi(\boldsymbol{x}) \in B$. Then obviously the map π is shift invariant, so that the space B with the hitting measure $\lambda = \pi(\mathbf{P})$ on it is necessarily a quotient of the Poisson boundary with respect to a certain G-invariant partition. Such quotients are called *μ-boundaries*. Of course, the topology on B is irrelevant, and any equivariant and shift invariant projection $\pi : (G^{\mathbb{Z}_+}, \mathbf{P}) \to (B, \lambda)$ gives rise to a μ-boundary.

The Poisson boundary is the *maximal* μ-boundary. Therefore, the problem of *identificating the Poisson boundary* of (G, μ) consists of two parts:

(1) To find (in geometric or combinatorial terms) a μ-boundary (B, λ);
(2) To show that this μ-boundary is maximal.

In other words, first one has to exhibit a certain system of invariants of stochastically significant behavior of sample paths at infinity, and then to show completeness of this system. A particular case is proving triviality of the Poisson boundary, i.e., proving maximality of the one-point μ-boundary.

We emphasize that even if a μ-boundary is realized on the boundary of a certain group compactification, the maximality of this μ-boundary has nothing to do with solvability of the Dirichlet problem for μ-harmonic functions with respect to this compactification.

The Poisson boundary is trivial for all measures on abelian and nilpotent groups; on the other hand, if the group G is non-amenable, then the Poisson boundary is non-trivial for any non-degenerate measure μ, see [Ka96] and references therein. For amenable groups one can always construct a measure μ with trivial Poisson boundary [KV83], [Ro81], but there may also be measures with a non-trivial boundary [Ka85a].

0.5. One can apply various direct methods of describing non-trivial behaviour of sample paths at infinity for finding a μ-boundary.

The following very useful idea of Furstenberg [Fu71] gives a general approach to constructing μ-boundaries. Let B be a separable *compact* G-space; by its compactness there exists a μ-stationary probability measure λ on B. Now, the Martingale Convergence Theorem implies that for a.e. sample path $\boldsymbol{x} = \{x_n\}$ the sequence of translations $x_n\lambda$ converges weakly to a measure $\lambda(\boldsymbol{x})$. Thus, the map $\boldsymbol{x} \mapsto \lambda(\boldsymbol{x})$ allows one to consider the space of probability measures on B as a μ-boundary. If the action of G on B has the property that for any non-atomic measure λ all weak limit points of the family of translations $\{g\lambda\}$, $g \in G$ are δ-measures (such actions are called μ-proximal [Fu73]), then almost all measures $\lambda(\boldsymbol{x})$ are δ-measures, so that (B, λ) is a μ-boundary.

If a group compactification $\overline{G} = G \cup \partial G$ has the property that $g_n\xi \to \gamma_+$ uniformly outside of every neighbourhood of γ_- in $\overline{G}$ whenever $g_n^{\pm 1} \to \gamma_\pm \in \partial G$, then the G-action on ∂G is mean proximal [Wo93] (see also [GM87]).

We introduce another condition inspired by the notion of "bilateral structures" playing an important role in this paper. If there exists a G-equivariant map S assigning to pairs of distinct points (γ_-, γ_+) from ∂G non-empty subsets ("strips") $S(\gamma_-, \gamma_+) \subset G$ such that for any distinct $\gamma_0, \gamma_1, \gamma_2 \in \partial G$ there are neighbourhoods $\mathcal{O}_o \subset \overline{G}$ and $\mathcal{O}_1, \mathcal{O}_2 \subset \partial G$ with $S(\gamma_-, \gamma_+) \cap \mathcal{O}_o = \varnothing$ for all points $\gamma_- \in \mathcal{O}_1, g_+ \in \mathcal{O}_2$ then the action of G on ∂G is mean proximal (Theorem 2.1.4).

Under either of these conditions n-fold convolutions μ_n of the measure μ weakly converge to the unique μ-stationary measure λ on ∂G. It makes this construction in a sense similar to the Patterson–Sullivan construction [Pa87], [Su79], the "geometry" of the group being determined by the choice of μ. However, an important difference is that in our case the measures μ_n are connected with the recurrence relation $\mu_{n+1} = \mu_n\mu = \mu\mu_n$, which provides the resulting boundary measure with new properties.

The hyperbolic compactification of *word hyperbolic groups* and the end compactification of *groups with infinitely many ends* satisfy both these conditions. Other examples where one can prove convergence of sample paths in an appropriate compactification and uniqueness of μ-stationary measures on the compactification boundary are *cocompact lattices in rank one Cartan–Hadamard manifolds* with respect to the visibility compactification [Ba89] and *mapping class groups* with respect to the *Thurston compactification* of Teichmüller space [KM96].

For a *semi-simple Lie group* $\mathcal{G}$ one has to consider the associated Riemannian symmetric space $S \cong \mathcal{G}/\mathcal{K}$, where $\mathcal{K}$ is a maximal compact subgroup. The boundary ∂S of the *visibility compactification* of S consists of $\mathcal{G}$-orbits ∂S_a parameterized by unit length vectors a from the closure of a dominant Weyl chamber $\mathfrak{A}^+$ in the Lie algebra of a Cartan subgroup $\mathcal{A}$. The orbits ∂S_a are isomorphic to the *Furstenberg boundary* $\mathcal{B} = \mathcal{G}/\mathcal{P}$ (here $\mathcal{P}$ is a minimal parabolic subgroup) for vectors a inside the Weyl chamber, and to quotients of $\mathcal{B}$ if a is degenerate [Ka89]. The Furstenberg boundary can be also defined as the space of asymptotic classes of Weyl chambers [Mo73] in complete analogy with the definition of the visibility boundary as the space of asymptotic classes of geodesic rays. For the group $SL(d, \mathbb{R})$ the Furstenberg boundary is the space of flags in $\mathbb{R}^d$ (the boundary circle of the hyperbolic plane if $d = 2$).

If a measure μ on $\mathcal{G}$ has a *finite first moment* $\int \mathrm{dist}(o, go)\, d\mu(g) < \infty$, then there exists a *Lyapunov vector* $a \in \overline{\mathfrak{A}}^+$ such that $r(x_n o)/n \to a$ for $\mathbf{P}$-a.e. sample path $\{x_n\}$ of the random walk (G, μ), where $r(x) \in \overline{\mathfrak{A}}^+$ is the *radial part* of a point $x \in S$

determined from the Cartan decomposition. If $a \neq 0$, then a.e. sequence $x_n o$ converges to the orbit ∂S_a [Ka89].

Embedding the symmetric space S into the space of probability measures on $\mathcal{B}$ by the map $go \mapsto gm$, where $o \cong \mathcal{K} \in S$, and m is the unique $\mathcal{K}$-invariant probability measure on $\mathcal{B}$, and taking closure in the weak topology gives rise to the *Satake–Furstenberg compactification* of S. Its boundary consists of several $\mathcal{G}$-transitive components, one of which (corresponding to limit δ-measures) is isomorphic to $\mathcal{B}$. Guivarc'h and Raugi [GR85] proved that if the measure μ satisfies certain non-degeneracy conditions (in particular, if the group generated by $\operatorname{supp}\mu$ is Zariski dense [GM89]), then a.e. sequence $x_n o$ converges in the Satake–Furstenberg compactification to $\mathcal{B}$. If the measure μ in addition has a finite first moment, then the μ-boundaries obtained by these two procedures are isomorphic, because the Lyapunov vector in this case is non-degenerate [GR85].

Realizing *non-compact spaces* as μ-boundaries in the case of Lie groups (or discrete subgroups of Lie groups) usually amounts to proving convergence in appropriate homogeneous spaces of the group by using contracting properties of the action and requires finiteness of the first moment of the measure μ, i.e., $\int \delta_K(g)\, d\mu(g) < \infty$, where $\delta_K(g) = \min\{n : g \in K^n\}$ is the word length on G determined by a compact symmetric neighbourhood of the identity [Az70], [Ra77], [Gu80a]. For example, let $G = \mathrm{Aff}(\mathbb{R}) = \{t \mapsto at + b, a \in \mathbb{R}_+, b \in \mathbb{R}\}$ be the *real affine group*. The finite first moment condition then takes the form $\int \big[|\log a(g)| + |(\log |b(g)|)|\big]\, d\mu(g) < \infty$. If $\alpha = \int \log a(g)\, d\mu(g) < 0$, then the elements $x_n = (a_n, b_n)$ of a.e. sample path act on $\mathbb{R}$ exponentially contracting, and looking at the formula for the group product in G one can immediately see that there exists a limit $b_\infty = \lim_{n\to\infty} b_n \in \mathbb{R}$. The same idea works for polycyclic groups or for discrete affine groups (Theorems 3.5.6, 3.6.4).

For discrete groups which are not immediately connected with Lie groups the variety of situations is wider and examples of non-trivial μ-boundaries realized on non-compact spaces and obtained from "elementary" probabilistic and combinatorial considerations include random walks on the *infinite symmetric group*, some *locally finite solvable groups*, and some *wreath products* [KV83], [Ka85a].

0.6. Two general ideas are very helpful for identification of the Poisson boundary of Lie groups. The first one is used for proving maximality of a given μ-boundary $Z = \pi(\Gamma)$. Suppose that a subgroup $H \subset G$ acts simply transitively on Z. If the fibers $\Gamma_z = \pi^{-1}(z)$ are non-trivial, then acting by H one extends a non-constant bounded function φ_z on Γ_z to a non-constant H-invariant function φ on Γ, which gives rise to a non-constant bounded H-invariant harmonic function. Thus, if one knows that the latter do not exist, then Z in fact coincides with the Poisson boundary. For an absolutely continuous measure μ on a non-compact semi-simple Lie group with finite center G this idea allowed Furstenberg [Fu63a] to identify the Poisson boundary with the Furstenberg boundary $\mathcal{B}$ of the corresponding symmetric space.

The other idea is used for finding out group elements $g \in G$ (*μ-periods*) such that their action on the Poisson boundary is trivial. If the sequence $(x_n^{-1} g x_n)$ has a limit point in G for a.e. path $\{x_n\}$, then g is a μ-period [Az70], [Gu73]. Applying these ideas (and with a heavy use of the structure theory of Lie groups) Azencott [Az70] and Raugi [Ra77] described the Poisson boundary for any spread out probability measure with a finite first moment on a connected Lie group G as a G-space determined by a family of cocycles associated with the measure μ.

For an illustration let us look again at the real affine group G. If $x_n = (a_n, b_n)$ and $g = (1, b)$, then $x_n^{-1} g x_n = (1, a_n^{-1} b)$. Thus, if $\alpha = \int \log a(g)\, d\mu(g) \ge 0$, then $H = \{(1, b)\}$ is a subgroup of the group of μ-periods, so that any bounded μ-harmonic function on G is H-invariant, i.e., depends on the component $a(g)$ only. The abelian group $\{(a, 0)\}$ does not have bounded harmonic functions, and the Poisson boundary of the random walk (G, μ) is thereby trivial. In the *contracting* case $\alpha < 0$, as we have already seen, $\mathbb{R}$ with the corresponding limit measure λ is a non-trivial μ-boundary. Since the subgroup H acts on $\mathbb{R}$ simply transitively, and there are no H-invariant bounded harmonic functions, the μ-boundary $(\mathbb{R}, \lambda)$ is maximal.

0.7. Yet another boundary associated with the random walk (G, μ) is the *Martin boundary* obtained by embedding the group G into the projective space of functions on G by using the Green kernel and taking the closure. The Martin boundary contains all minimal positive harmonic functions, and any positive harmonic function can be uniquely decomposed as an integral of minimal ones. Considered as a measure space with the representing measure of the function $\mathbf{1}$, the Martin boundary is isomorphic to the Poisson boundary, see [Ka96] and references therein. Thus, a description of the Martin boundary would imply a description of the Poisson boundary. However, there is a fundamental difference between the Poisson and the Martin boundaries: the former is a measure space, whereas the latter is a topological space.

The most general approach to the description of the Martin boundary belongs to Ancona [An87], [An90], and is a far reaching generalization of earlier results for *free* and *Fuchsian groups* [DM61], [LM71], [De75], [Se83], for *trees* [PW87], [CSW93] and for the Brownian motion on *Cartan–Hadamard manifolds* with pinched sectional curvatures [AS85]. He showed that for a large class of "local" Markov operators (diffusion ones in the continuous setup and finite range ones in discrete situations) on *Gromov hyperbolic spaces* the Green kernel is almost multiplicative along geodesics, which implies that the Martin compactification coincides with the hyperbolic compactification. In particular, the Martin boundary for all finitely supported measures on hyperbolic groups is the hyperbolic boundary.

The "locality" assumption is crucial for the Martin boundary methods. For example, it is unclear whether Ancona's technique works for hyperbolic groups when the measure μ has a "very fast" decay at infinity, instead of being finitely supported. Moreover, the Martin boundary is "less functorial" (see [Ka92]) and "less stable" than the Poisson boundary. A recent example of Ballmann and Ledrappier [BL96] shows that there is a probability measure with a finite first logarithmic moment on a free group such that the Martin boundary of the corresponding random walk is homeomorphic to the circle and not to the space of ends (although from the measure theoretical point of view the Poisson boundary can be still identified with the space of ends).

0.8. In the present paper we are addressing the problem of identification of the Poisson boundary for random walks on a discrete group G under fairly mild conditions on decay of the measure μ at infinity (a finite first moment is sufficient). The methods used for Lie groups or the Martin theory methods are not applicable in this situation. The notion of *entropy* in explicit [Av72], [Av76], [KV83], [De80], [De86] or implicit form (via differential entropy [Fu71], asymptotic growth [Gu80b], Hausdorff dimension [Le83], [Le85], [BL94]) turned out to be much more efficient for dealing with the Poisson boundary of random walks on discrete groups.

We develop here a new method based on estimating the *entropy of conditional random walks*, which incorporates and generalizes all these approaches. Instead of using structure theory this method relies upon volume estimates for random walks and it is applicable both to discrete and continuous groups. It leads to two simple purely geometric criteria of boundary maximality. These criteria bear hyperbolic nature and allow us to identify the Poisson boundary with natural boundaries for several classes of groups with "hyperbolic properties": word hyperbolic groups (more generally, discrete groups of isometries of Gromov hyperbolic spaces), groups with infinitely many ends, cocompact lattices in Cartan–Hadamard manifolds, discrete subgroups of semi-simple Lie groups, polycyclic groups and some other semi-direct and wreath products. This is the main result of the present paper. Partial announcements were made in the author's notes [Ka85b], [Ka94].

Let μ be a probability measure on a countable group G with finite *entropy* $H(\mu) = -\sum \mu(g)\log\mu(g)$. If G is a finitely generated group, and the measure μ has a finite first moment in G, then its entropy is also finite. The limit $h(G,\mu) = \lim H(\mu_n)/n$ of normalized entropies of n-fold convolutions of μ is called the *entropy of the random walk* (G,μ) [Av72], [KV83], [De86]. As it follows from the Kingman Subadditive Ergodic Theorem, the entropy $h(G,\mu)$ coincides with the asymptotic entropy $\mathbf{h}(\mathbf{P})$ of the measure $\mathbf{P}$ in the path space $G^{\mathbb{Z}_+}$ in the following sense: the one-dimensional distributions μ_n of the measure $\mathbf{P}$ have the property that $-\log\mu_n(x_n)/n \to h(G,\mu)$ for $\mathbf{P}$-a.e. $\boldsymbol{x} = \{x_n\} \in G^{\mathbb{Z}_+}$ and in the space $L^1(\mathbf{P})$.

The Poisson boundary of (G,μ) is trivial iff $h(G,\mu) = 0$ [De80], [KV83]. It turns out that this criterion can be generalized to a criterion of maximality of a given μ-boundary (B,λ), which is formulated in terms of *conditional walks* associated with points $b \in B$. The conditional measures $\mathbf{P}^b$, $b \in B$ are the measures in the path spaces of Markov chains with transition probabilities $p^b(x,y) = \mu(x^{-1}y)dy\lambda/dx\lambda(b)$. Then, for a given μ-boundary (B,λ) there exists a number $E(B,\lambda)$ such that for λ-a.e. point $b \in B$ the asymptotic entropy of the measure $\mathbf{P}^b$ exists and equals $\mathbf{h}(\mathbf{P}^b) = h(G,\mu) - E(B,\lambda)$, and (B,λ) is maximal iff $E(B,\lambda) = h(G,\mu)$ (Theorems 1.7.5, 1.7.6). Thus, a μ-boundary (B,λ) is maximal iff the asymptotic entropies of all conditional measures $\mathbf{P}^b$, $b \in B$ vanish.

0.9. Now we can formulate two simple geometric criteria of maximality of a μ-boundary for a measure μ with finite entropy. Both require an approximation of the sample paths of the random walk in terms of their limit behaviour. For simplicity we assume that G is finitely generated, and denote by $d(g_1,g_2) = \delta(g_1^{-1}g_2)$ the left-invariant metric on G corresponding to a word length δ. Let $(B,\lambda) \cong (\Gamma_\xi,\nu_\xi)$ be a μ-boundary presented as the quotient of the Poisson boundary (Γ,ν) by a certain measurable G-invariant partition ξ, and $\mathbf{bnd}_\xi : (G^{\mathbb{Z}_+},\mathbf{P}) \to (\Gamma,\nu) \to (\Gamma_\xi,\nu_\xi) \cong (B,\lambda)$ be the corresponding projection from the path space onto (B,λ).

The first criterion says that if there is a family of measurable maps $\pi_n : \Gamma \to G$ such that a.e. $d(x_n, \pi_n(\mathbf{bnd}_\xi\, \boldsymbol{x})) = o(n)$, then (B,λ) is maximal ("*ray*", or, "*unilateral*" approximation, Theorem 2.3.2). This is an immediate corollary of Theorem 1.7.6.

The second criterion applies simultaneously to a μ-boundary (B_+,λ_+) and to a $\check\mu$-boundary (B_-,λ_-) (where $\check\mu(g) = \mu(g)^{-1}$ is the *reflected measure* of μ). Denote by B_n the balls of the word metric centered at e. If there is a G-equivariant measurable map

S assigning to pairs $(b_-,b_+)\in B_-\times B_+$ non-empty subsets $S(b_-,b_+)\subset G$ such that for a.e. $(b_-,b_+)\in B_-\times B_+$

$$\frac{1}{n}\log\bigl|S(b_-,b_+)\cap B_{\delta(x_n)}\bigr|\to 0 \tag{0.8}$$

in probability with respect to the measure $\mathbf{P}$, then both boundaries (B_-,λ_-) and (B_+,λ_+) are maximal ("*strip*", or, "*bilateral*" approximation, Theorem 2.4.5). This criterion is inspired by the use of bilateral geodesics in cocompact rank 1 Cartan–Hadamard manifolds by Ledrappier and Ballmann [BL94].

The proof of the second criterion makes use of the space $(G^{\mathbb{Z}},\overline{\mathbf{P}})$ of *bilateral* paths $\{x_n\}, n\in\mathbb{Z}$ of the random walk (G,μ) passing through the identity e at time 0. This space is isomorphic to the space of bilateral sequences of independent μ-distributed increments $\{h_n\}, n\in\mathbb{Z}$ under the map $x_n=x_{n-1}h_n$, and is decomposable into a product of unilateral path spaces of the random walks $(G,\check\mu)$ and (G,μ) corresponding to negative and positive times n, respectively. The bilateral Bernoulli shift in the space of increments induces then an ergodic measure preserving transformation $\overline{U}$ of $(G^{\mathbb{Z}},\overline{\mathbf{P}})$. Denote by $\mathbf{bnd}_\pm$ the projections from $(G^{\mathbb{Z}},\overline{\mathbf{P}})$ onto the boundaries $(B_\pm,\lambda_\pm)$. Then $\mathbf{bnd}_\pm(\overline{U}^n\boldsymbol{x})=x_n^{-1}\mathbf{bnd}_\pm\boldsymbol{x}$, so that by equivariance of the strip map S for any $n\in\mathbb{Z}$

$$\overline{\mathbf{P}}\bigl[x_n\in S(\mathbf{bnd}_-\boldsymbol{x},\mathbf{bnd}_+\boldsymbol{x})\bigr]=\overline{\mathbf{P}}\bigl[e\in S(\mathbf{bnd}_-\boldsymbol{x},\mathbf{bnd}_+\boldsymbol{x})\bigr]=p\,.$$

Since the strips $S(b_-,b_+)$ are a.e. non-empty, we may assume that $p>0$, so that sample paths of the conditional walk conditioned by $b_+\in B_+$ belong to $S(b_-,b_+)$ with probability p, which implies that the asymptotic entropy of the corresponding conditional measure $\mathbf{P}^{b_+}$ must be zero.

Subexponentiality of the intersections $\bigl[S(b_-,b_+)\cap B_{\delta(x_n)}\bigr]$ is the key condition here. Thus, the "thinner" are the strips $S(b_-,b_+)$ themselves, the larger is the class of measures for which condition (0.8) is satisfied, i.e., sample paths $\{x_n\}$ may be allowed to go to infinity "faster". If the strips $S(\gamma_-,\gamma_+)$ grow *subexponentially* then condition (0.8) is satisfied for any probability measure μ with a *finite first moment*, and if the strips grow *polynomially* then (0.8) is satisfied for any measure μ with a finite *first logarithmic moment* $\sum\log\delta(g)\mu(g)$ (Theorem 2.4.6).

0.10. The "ray criterion" provides more information than the "strip criterion" about the behaviour of sample paths of the random walk (which can be also helpful for other issues than just identification of the Poisson boundary; e.g., see [Le97], [Ka97] where it is used for estimating the Hausdorff dimension of the harmonic measure). On the other hand, for checking the ray criterion one often needs rather elaborate estimates, whereas existence of strips is usually almost evident, and estimates of their growth are not very hard. Let us look at how we use the ray and the strip approximation criteria for identificating the Poisson boundary of concrete groups.

For *word hyperbolic groups* (more generally, discontinuous groups of isometries of Gromov hyperbolic spaces) the ray criterion for measures μ with a finite first moment amounts to proving that for any sequence x_n in the group such that $d(x_0,x_n)/n\to l>0$ and $d(x_n,x_{n+1})=o(n)$ there exists a geodesic ray α with $d(x_n,\alpha(ln))=o(n)$

(Theorem 3.1.5). This is a purely geometric property of Gromov hyperbolic spaces (cf. below an analogous property of Riemannian symmetric spaces, Theorem 3.4.3), which, nevertheless, is not totally obvious. On the other hand, the strip $S(\xi_-, \xi_+)$ corresponding to a pair of points from the hyperbolic boundary is naturally defined as a union of all geodesics with endpoints ξ_-, ξ_+ and has a linear growth. It implies that the Poisson boundary for any measure μ with finite entropy and a finite first logarithmic moment on a hyperbolic group identifies with the hyperbolic boundary (Theorem 3.1.10).

In the case of *groups with infinitely many ends* obtaining a ray approximation becomes more difficult. However, once again, defining appropriate strips associated with pairs of distinct ends ω_-, ω_+ presents no difficulty: take for $S(\omega_-, \omega_+)$ the union of all R-balls separating ω_- and ω_2, where $R = R(\omega_-, \omega_+)$ is the minimal number for which such balls exist. It enables us to identify the Poisson boundary with the space of ends under the same conditions as for hyperbolic groups (Theorem 3.2.5). Note that this approach does not use at all the structure theory of groups with infinitely many ends and appeals directly to their definition.

The only geometric property of hyperbolic groups and groups with infinitely many ends and their respective compactifications used here is that for any two boundary points $\xi_- \neq \xi_+$ the pencil $P(\xi_-, \xi_+)$ of infinite geodesics with limit points ξ_-, ξ_+ is non-empty and there exists a finite set $A(\xi_-, \xi_+)$ such that any geodesic from $P(\xi_-, \xi_+)$ intersects $A(\xi_-, \xi_+)$. It turns out that any group compactification $\overline{G} = G \cup \partial G$ with this property is maximal form a measure theoretical point of view. Namely, if μ is a probability measure on G with a finite first moment, then there exists a unique μ-stationary measure λ on ∂G, and the measure space $(\partial G, \lambda)$ is the Poisson boundary of (G, μ) (Theorem 2.4.7).

In the next two examples the ray approximation fails completely, but geodesics in the corresponding enveloping spaces still easily provide us with linear growth strips.

For *cocompact lattices in rank one Cartan–Hadamard manifolds* for applying the strip criterion one takes geodesics joining pairs of points from the visibility boundary (which a.e. exist due to a result of Ballmann [Ba89]). Once again, the very existence of such geodesics implies that the Poisson boundary coincides with the visibility boundary for all measures μ with finite entropy and first logarithmic moment (Theorem 3.3.2). Together with a description of the Poisson boundary for discrete subgroups of semi-simple Lie groups (see below, Theorems 3.4.6, 3.4.8), and taking into account the Rank Rigidity Theorem [Ba95], it allows us to identify the Poisson boundary for all *fundamental groups of compact non-positively curved Riemannian manifolds*.

The *mapping class groups* are treated in a separate joint paper with Masur [KM96]. In this case the strips are defined by using Teichmüller geodesic lines in Teichmüller space associated with any two distinct uniquely ergodic projective measured foliations, which implies identification of the Poisson boundary with a natural geometric boundary (the boundary of the Thurston compactification) for all measures with finite entropy and finite first logarithmic moment.

For *discrete subgroups of semi-simple Lie groups* the difference between using convergence in the Satake–Furstenberg and visibility compactifications for identifying the Poisson boundary is in a trade-off between the moment and irreducibility conditions.

Depending on situation, one can use either of these compactifications for describing the Poisson boundary by applying the corresponding geometric criterion. If μ is a non-degenerate measure on a *Zariski dense* discrete subgroup G with *finite first logarithmic moment* $\sum \log \mathrm{dist}(o, go)\mu(g)$ and finite entropy, then irreducibility of the harmonic measures of μ and $\check\mu$ on the Furstenberg boundary $\mathcal{B}$ allows one to assign to a.e. pair of points in $\mathcal{B}$ a uniquely determined *flat* in S; since flats have polynomial growth, by using the strip criterion we obtain that $\mathcal{B}$ is the Poisson boundary of the measure μ (Theorem 3.4.8).

For an *arbitrary* discrete subgroup G provided the measure μ has *finite first moment* $\sum \mathrm{dist}(o, go)\mu(g)$, the sequence $x_n o$ is a.e. *regular* in the sense that there exist a geodesic ray ξ such that $\mathrm{dist}(x_n o, \xi(n\|a\|)) = o(n)$, where a is the Lyapunov vector. This fact is a geometric counterpart of the Oseledec Multiplicative Ergodic Theorem [Ka89], and in view of the ray criterion it immediately implies identification of the Poisson boundary with the corresponding orbit $\partial S_{a/\|a\|}$ if $a \neq 0$ (Theorem 3.4.6). If the Lyapunov vector vanishes, then the Poisson boundary is trivial.

A *polycyclic group* G up to a semi-simple splitting is a semi-direct product $A \leftthreetimes N$ of two torsion free finitely generated group: abelian A and nilpotent N. Let μ be a measure with a finite first moment on G. The barycenter of the projection of μ to A determines an automorphism T_μ of $\mathcal{N}$ (the Lie hull of N) which gives rise to a decomposition of $\mathcal{N}$ into contracting $\mathcal{N}_-$, neutral $\mathcal{N}_0$ and expanding $\mathcal{N}_+$ subgroups. The homogeneous space $A \leftthreetimes \mathcal{N}/A\mathcal{N}_0\mathcal{N}_+$ (identified with the contracting subgroup $\mathcal{N}_-$) is a μ-boundary, and the expanding subgroup $\mathcal{N}_+$ (i.e., the contracting subgroup for the reflected measure $\check\mu$) is a $\check\mu$-boundary (cf. the example above with the affine group). Any pair of points from $\mathcal{N}_-$ and $\mathcal{N}_+$ determines (as intersection of the corresponding cosets) a coset of $A\mathcal{N}_0$ in $A \leftthreetimes \mathcal{N}$, which gives rise to equivariant strips in G. Showing that these strips are "thin enough" (here we have to use a special metric on G with infinite balls) boils down to an easy estimate of the growth of the neutral component along sample paths of the random walk, and the Poisson boundary of (G, μ) identifies with the contracting subgroup $\mathcal{N}_-$ (Theorem 3.5.6).

For a general *semi-direct product* $G = A \leftthreetimes H$ any measurable H-equivariant map $\pi : B_- \times B_+ \to H$ determines equivariant strips $S(b_-, b_+)$ with the same growth as A. In particular, if the measure μ on G has a finite first moment and the growth of A is subexponential, then very existence of π implies maximality of the boundaries $(B_\pm, \lambda_\pm)$ (Theorem 3.6.2).

The *Baumslag–Solitar group* $G = BS(1,p) = \langle a, b | aba^{-1} = b^p \rangle \cong \mathrm{Aff}(\mathbb{Z}[\frac{1}{p}])$ is isomorphic to the semi-direct product $\mathbb{Z} \leftthreetimes \mathbb{Z}[\frac{1}{p}]$ determined by the action $T^z f = p^z f$. It has two boundaries ("lower" and "upper") $\mathbb{R}$ and $\mathbb{Q}_p$ obtained by completing $\mathbb{Z}[\frac{1}{p}]$ in the usual and in the "p-adic" (p is not necessarily a prime) metrics [KV83], [FM97]. If μ is a measure with a finite first moment on G, denote by $\overline{\mu}_{\mathbb{Z}}$ the mean of its projection $\mu_{\mathbb{Z}}$ to $\mathbb{Z}$. The Poisson boundary of (G, μ) is then determined by the sign of $\overline{\mu}_{\mathbb{Z}}$. Namely, if $\overline{\mu}_{\mathbb{Z}} = 0$, then the Poisson boundary is trivial [KV83]. If $\overline{\mu}_{\mathbb{Z}} < 0$ (resp., > 0), then the lower boundary $\mathbb{R}$ (resp., the upper boundary $\mathbb{Q}_p$) is a non-trivial μ-boundary (cf. the example with the real affine group). The map $\pi(x, \xi) = x + \{\{\xi\} - \{x\}\}$ from $\mathbb{R} \times \mathbb{Q}_p$ to $\mathbb{Z}[\frac{1}{p}]$ is $\mathbb{Z}[\frac{1}{p}]$-equivariant (here $x \mapsto \{x\}$ is the function assigning to a real or p-adic number its fractional part $0 \le \{x\} < 1$). Thereby, the lower boundary $\mathbb{R}$ (if $\overline{\mu}_{\mathbb{Z}} < 0$) or the upper boundary $\mathbb{Q}_p$ (if $\overline{\mu}_{\mathbb{Z}} > 0$) are maximal (Theorem 3.6.4).

In the same way we obtain maximality of natural μ-boundaries for *wreath products* $G = A \leftthreetimes \mathbf{fun}(A,B)$, where $\mathbf{fun}(A,B)$ is the group of all finitely supported B-valued configurations on A. If the group A has subexponential growth, the measure μ on G has a finite first moment, and there exists a homomorphism $\psi : A \to \mathbb{Z}$ such that the mean $\overline{\mu}_{\mathbb{Z}}$ of the measure $\mu_{\mathbb{Z}} = \psi(\mu)$ is non-zero, then for $\mathbf{P}$-a.e. sample path $\{(x_n, \varphi_n)\}$ the configurations φ_n converge pointwise to a limit configuration $\lim \varphi_n$ from the group $\mathbf{Fun}(A,B)$ of *all* B-valued configurations on A, and the Poisson boundary of the pair (G,μ) is isomorphic to $\mathbf{Fun}(A,B)$ with the resulting limit measure λ (Theorem 3.6.6). A particular case are the so-called *groups of dynamical configurations*, or *lamplighter groups* $G_k = \mathbb{Z}^k \leftthreetimes \mathbf{fun}(\mathbb{Z}^k, \mathbb{Z}_2)$ first considered in [KV83].

As an application we obtain that if μ is a probability measure with a finite first moment on a finitely generated group G of subexponential growth, and there exists a homomorphism $\psi : G \to \mathbb{Z}$ such that the mean $\overline{\mu}_{\mathbb{Z}}$ of the measure $\mu_{\mathbb{Z}} = \psi(\mu)$ is non-zero, then the *exchangeable σ-algebra* of the random walk (G,μ) is described by the final occupation times (Theorem 3.6.10).

0.11. Moment conditions (finite first moment moment of finite first logarithmic moment) and, in the first place, finiteness of entropy are crucial for the methods used in the present paper, and the question about maximality of natural μ-boundaries for an *arbitrary* measure μ, say, on a word hyperbolic groups (just on a free group, to take the simplest case), or on Zariski dense discrete subgroups of semi-simple Lie groups remains open.

On the other hand, our methods could be also applied in the continuous situations. The entropy approach was used for finding out when the Poisson boundary is trivial for random walks with absolutely continuous measure μ on general locally compact groups (in particular, Lie groups) [Av76], [Gu80a], [De86], [Va86], [Al87]. Here one should replace the entropy $H(\mu)$ with the *differential entropy*

$$H_{diff}(\mu) = -\int \log \frac{d\mu}{dm}(g)\, d\mu(g) ,$$

where m is the left Haar measure on G. Likewise, our entropy criterion of maximality of μ-boundaries in terms of entropy of conditional random walks can be also extended to continuous groups, which leads to analogous "ray" and "strip" approximation geometric criteria applicable to all spread out measures μ with a finite first moment. It gives a unified approach to discrete and continuous situations. For example, for the real affine group Aff($\mathbb{R}$) taking for strips the sets $A_\gamma = \{(a,b) : b = \gamma\}$ (i.e., the hyperbolic geodesics joining points from the boundary of the upper half-plane with the point at infinity) shows at once that the Poisson boundary coincides with $\mathbb{R}$ in the contracting case $\alpha < 0$ and is trivial in the expanding case $\alpha > 0$. This is the same idea that we have used for the Baumslag–Solitar group $BS(1,p)$, and it also works for the affine group of homogeneous trees [CKW94].

Coming back to our point of departure, the classical Poisson formula for bounded harmonic functions on the hyperbolic plane, we may conclude that our methods also shed a new light on its nature. Namely, the fact that the isometry between the space $C(\partial\mathbb{D})$ and the space of harmonic functions continuous up to the boundary extends to an isometry between the space $L^\infty(\partial\mathbb{D})$ and the space of all bounded harmonic functions can be explained just by existence of infinite geodesics joining pairs of distinct boundary points. We shall return to this subject elsewhere.

0.12. The paper consists of three major parts. In the first part we introduce random walks on groups (Section 1.1), define the Poisson boundary (Section 1.2), and prove the Poisson formula (Section 1.3). Further we discuss the notion of a μ-boundary, formulate the problem of identification of the Poisson boundary and obtain a conditional decomposition of the measure in the path space of the original random walk with respect to a given μ-boundary (Section 1.4). In Section 1.5 we prove coincidence of the Poisson and the tail boundaries, which is the key ingredient of the entropy theory of random walks described in Section 1.6. Finally, in Section 1.7 we obtain a measure theoretic criterion of maximality of a μ-boundary in terms of entropies of conditional random walks.

The second part is devoted to geometric criteria of boundary maximality. We begin with studying relationships between group compactifications and μ-boundaries and obtaining conditions for realizing the boundary of a given compactification as a μ-boundary (Section 2.1). Then after discussing various notions of measuring "size" and "length" in groups (Section 2.2) we prove geometric criteria of boundary maximality in terms of the ray approximation (Section 2.3) and the strip approximation (Section 2.4). The latter can be also reformulated using the notion of asymptotically dissipative group actions (Section 2.5).

In the final third part we apply general criteria to concrete classes of groups, and describe the Poisson boundary for word hyperbolic groups (Section 3.1), groups with infinitely many ends (Section 3.2), fundamental groups of compact rank 1 Cartan–Hadamard manifolds (Section 3.3), discrete subgroups of semi-simple Lie groups (Section 3.4), polycyclic groups (Section 3.5), and some wreath and semi-direct products including Baumslag–Solitar groups $BS(1,p)$ and lamplighter groups (Section 3.6).

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1. Entropy of random walks

1.1. Random walks on groups.

1.1.1. Let G be a countable group, and μ – a probability measure on G. We shall denote by $\operatorname{sgr}(\mu)$ (resp., $\operatorname{gr}(\mu)$) the semigroup (resp., the group) generated by the support $\operatorname{supp}\mu$ of the measure μ.

Definition. The (right) *random walk* on G determined by the measure μ is the Markov chain on G with the transition probabilities

$$p(x,y) = \mu(x^{-1}y) \tag{1.1.1}$$

invariant with respect to the left action of the group G on itself.

Thus, the position x_n of the random walk at time n is obtained from its position x_0 at time 0 by multiplying by independent μ-distributed right *increments* h_i:

$$x_n = x_0 h_1 h_2 \cdots h_n \ , \tag{1.1.2}$$

and the set of all points in G attainable by the random walk from the identity e is the semigroup $\operatorname{sgr}(\mu)$.

1.1.2. The *Markov operator* $P = P_\mu$ of averaging with respect to the transition probabilities of the random walk (G, μ) is

$$P_\mu f(x) = \sum_y p(x,y) f(y) = \sum_h \mu(h) f(xh) \ .$$

Its adjoint operator acts on the space of measures on G by the formula

$$\theta P(y) = \sum_x \theta(x) p(x,y) = \sum_x \theta(x) \mu(x^{-1}y) = \theta\mu(y) \ . \tag{1.1.3}$$

If θ is the distribution of the position of the random walk at time n, then $\theta P = \theta\mu$ is the distribution of its position at next time $n+1$.

> *Here and below we use the notation $\alpha\beta$ to denote the convolution of a measure α on G and a measure β on a G-space X (or, on the group G itself), i.e., the image of the product measure $\alpha \otimes \beta$ under the map $(g,x) \mapsto gx$.*

1.1.3. Denote by $G^{\mathbb{Z}_+}$ the space of *sample paths* $\boldsymbol{x} = \{x_n\}$, $n \ge 0$ endowed with the coordinate-wise action of G. *Cylinder* subsets of the path space are denoted

$$C_{g_0, g_1, \dots, g_n} = \{\boldsymbol{x} \in G^{\mathbb{Z}_+} : x_i = g_i, \ 0 \le i \le n\} = \bigcap_{i=0}^n C^i_{g_i} \ , \tag{1.1.4}$$

where $C^i_g = \{\boldsymbol{x} \in G^{\mathbb{Z}_+} : x_i = g\}$ are the *one-dimensional cylinders*.

1.1.4. An initial distribution θ on G determines the *Markov measure* $\mathbf{P}_\theta$ in the path space. It is the isomorphic image of the measure $\theta \otimes \bigotimes_{n=1}^\infty \mu$ under the map (1.1.2), in other words, for any cylinder set (1.1.4)

$$\mathbf{P}_\theta(C_{g_0, g_1, \dots, g_n}) = \theta(g_0)\mu(g_0^{-1}g_1) \dots \mu(g_{n-1}^{-1}g_n) \ . \tag{1.1.5}$$

The *one-dimensional distribution* of the measure $\mathbf{P}_\theta$ at time n (i.e., its image under the projection $\boldsymbol{x} \mapsto x_n$) is $\theta P^n = \theta\mu_n$, where μ_n is the *n-fold convolution* of the measure μ.

If θ is the unit mass at a point $g \in G$, then the corresponding measure in the path space is denoted $\mathbf{P}_g$. By $\mathbf{P} = \mathbf{P}_e$ we denote the measure in the path space corresponding to the initial distribution concentrated at the group identity e (this is the most important for us measure in the path space). Then for an arbitrary initial distribution θ

$$\mathbf{P}_\theta = \sum \theta(g)\mathbf{P}_g = \sum \theta(g) g\mathbf{P} = \theta\mathbf{P} \ . \tag{1.1.6}$$

Being isomorphic to a countable product of discrete measure spaces, the path space $(G^{\mathbb{Z}_+}, \mathbf{P}_\theta)$ is a *Lebesgue space*, which allows us to use in the sequel the standard ergodic theory technique of *measurable partitions* and *conditional measures* due to Rokhlin (e.g., see [CFS82]).

1.2. The Poisson boundary.

1.2.1. Let $T:\{x_n\}\mapsto\{x_{n+1}\}$ be the *time shift* in the path space $G^{\mathbb{Z}_+}$. Then by (1.1.3) and (1.1.5)

$$T\mathbf{P}_\theta=\mathbf{P}_{\theta P}=\mathbf{P}_{\theta\mu} \tag{1.2.1}$$

for an arbitrary initial distribution θ on G, so that all measures $\mathbf{P}_\theta$ with $\operatorname{supp}\theta=G$ are quasi-invariant with respect to T. Since the counting measure m on G is obviously *stationary* with respect to the operator P (i.e., $mP=m$), the σ-finite measure $\mathbf{P}_m$ is T-invariant.

1.2.2. Definition. The space of ergodic components Γ of the time shift T in the path space $(G^{\mathbb{Z}_+},\mathbf{P}_m)$ is called the *Poisson boundary* of the random walk (G,μ).

In a more detailed way, denote by $\sim$ the *orbit equivalence relation* of the shift T on the path space $G^{\mathbb{Z}_+}$:

$$\boldsymbol{x}\sim\boldsymbol{x}' \iff \exists\, n,n'\ge 0: T^n\boldsymbol{x}=T^{n'}\boldsymbol{x}'\;. \tag{1.2.2}$$

This orbit equivalence relation is also sometimes called *grand* or *asynchronous* to distinguish it from another equivalence relation associated with orbits of T; see below **1.5.1**. Denote by $\mathcal{A}_T$ the σ-algebra of all measurable unions of $\sim$-classes (mod 0) in the space $(G^{\mathbb{Z}_+},\mathbf{P}_m)$, i.e., the σ-algebra of all T-invariant sets (mod 0). Since $(G^{\mathbb{Z}_+},\mathbf{P}_m)$ is a Lebesgue space, there is a (unique up to an isomorphism) measurable space Γ (the *space of ergodic components*) and a map $\mathbf{bnd}: G^{\mathbb{Z}_+}\to\Gamma$ such that the σ-algebra $\mathcal{A}_T$ coincides (mod 0) with the σ-algebra of $\mathbf{bnd}$-preimages of measurable subsets of Γ (see also [KP72], [Sc78] for a construction of the ergodic decomposition of a measure type preserving action of a countable group). Denote by η the corresponding measurable partition of the path space into $\mathbf{bnd}$-preimages of points from Γ, i.e., the *measurable envelope* of the equivalence relation $\sim$. We shall call η the *Poisson partition*.

1.2.3. Definition. For an initial probability distribution θ on G the measure $\nu_\theta=\mathbf{bnd}(\mathbf{P}_\theta)$ is called the *harmonic measure* determined by θ.

The measure type $[\nu_m]$ on Γ which is the image of the type of the measure $\mathbf{P}_m$ is called the *harmonic measure type*. In other words, $[\nu_m]$ is the type of all measures $\mathbf{bnd}\,\mathbf{P}_\theta$, where θ is a finite measure on G equivalent to m (the measure $\mathbf{bnd}\,\mathbf{P}_m$ itself is trivially infinite). Any harmonic measure is absolutely continuous with respect to the harmonic measure type (but not necessarily belongs to it, see below Example 1.2.9).

1.2.4. By definition of Γ as the space of ergodic components of the shift T, for an arbitrary initial distribution θ we have $\mathbf{bnd}(\mathbf{P}_\theta)=\mathbf{bnd}(T\mathbf{P}_\theta)$, so that by (1.2.1)

$$\nu_\theta=\mathbf{bnd}(\mathbf{P}_\theta)=\mathbf{bnd}(T\mathbf{P}_\theta)=\mathbf{bnd}(\mathbf{P}_{\theta P})=\nu_{\theta P}=\nu_{\theta\mu}\;. \tag{1.2.3}$$

1.2.5. The coordinate-wise action of G on the path space commutes with the shift T, hence it projects to a canonical G-action on Γ (because the orbit equivalence relation

$\sim$ is G-invariant). By G-invariance of the measure m, the harmonic measure type is quasi-invariant with respect to the action of G (i.e., any G-translation of any null set of $[\nu_m]$ is also a null set of $[\nu_m]$).

Denote by $\nu = \nu_e = \mathbf{bnd}\,(\mathbf{P})$ the harmonic measure of the group identity. Then by (1.1.6) for an arbitrary initial distribution θ

$$\nu_\theta = \mathbf{bnd}\,(\mathbf{P}_\theta) = \mathbf{bnd}\,(\theta\,\mathbf{P}) = \theta\,\mathbf{bnd}\,(\mathbf{P}) = \theta\nu\ . \tag{1.2.4}$$

In view of (1.2.3), it implies

Proposition. *The harmonic measure $\nu = \nu_e$ is μ-stationary, i.e.,*

$$\nu = \mu\nu = \sum_g \mu(g) g\nu\ . \tag{1.2.5}$$

Remark. Formula (1.2.5) implies that $g\nu \prec \nu$ for all $g \in \mathrm{sgr}\,(\mu)$. Therefore, if $\mathrm{sgr}\,(\mu) = G$, then the measure ν is quasi-invariant and belongs to the harmonic measure type $[\nu_m]$. However, this is not necessarily so under the weaker assumption $\mathrm{gr}\,(\mu) = G$, see Example 1.2.9.

1.2.6. The Bernoulli shift in the space of increments of the random walk determines the measure preserving ergodic transformation

$$(U\boldsymbol{x})_n = x_1^{-1} x_{n+1} \tag{1.2.6}$$

of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$. Since the paths $\boldsymbol{x}$ and $x_1(U\boldsymbol{x})$ are $\sim$-equivalent, we have

Lemma. *For $\mathbf{P}$-a.e. sample path $\boldsymbol{x} = \{x_n\} \in G^{\mathbb{Z}_+}$*

$$\mathbf{bnd}\,\boldsymbol{x} = x_1 \mathbf{bnd}\,U\boldsymbol{x}\ .$$

1.2.7. Below we shall be interested in describing the Poisson boundary for the initial distribution δ_e, i.e., in describing the measure space (Γ, ν). Fixing the harmonic measure ν on Γ makes the Poisson boundary a canonically defined measure space endowed with an action of the semigroup $\mathrm{sgr}\,(\mu)$ (see **1.2.5**, **1.2.6**).

Although in general the measure ν does not have to belong to the harmonic measure type $[\nu_m]$, the Poisson boundary (Γ, ν_θ) for an arbitrary initial distribution θ can be recovered from the space (Γ, ν) in virtue of formula (1.2.4). The only minor difficulty here is that the measure space (Γ, ν) is acted upon by the semigroup $\mathrm{sgr}\,(\mu)$ only. In order to obtain the Poisson boundary (Γ, ν_θ) one then has to take the quotient of the product space $(G \times \Gamma, \theta \otimes \nu)$ with respect to the equivalence relation obtained by identifying pairs $(g_1, g_2\gamma)$ and $(g_1 g_2, \gamma)$ for all $g_1 \in G, g_2 \in \mathrm{sgr}\,(\mu)$ and ν-a.e. $\gamma \in \Gamma$. One can easily see that if the harmonic measure ν is concentrated on a single point, then the Poisson boundary (Γ, ν_θ) is just the quotient space $(G, \theta)/\mathrm{gr}\,(\mu)$. In particular, if $\mathrm{gr}\,(\mu) = G$, then triviality of the harmonic measure ν is equivalent to triviality of the harmonic measure type $[\nu_m]$.

1.2.8. Triviality of the Poisson boundary (Γ,ν) is equivalent to the property

$$\lim_{n\to\infty}\frac{1}{n}\Big\|\sum_{k=1}^{n}\mu_n - g\sum_{k=1}^{n}\mu_n\Big\| \qquad \forall\, g\in \mathrm{sgr}\,(\mu)\,,$$

i.e., to strong convergence of the sequence of Cesaro averages of the convolutions μ_n to a left-invariant mean on $\mathrm{gr}\,(\mu)$ [KV83], [Ka92]. Thus, if $\mathrm{gr}\,(\mu)$ is non-amenable, then (Γ,ν) is necessarily non-trivial.

For any amenable group G there exists a measure μ with $\mathrm{supp}\,\mu = G$ such that its Poisson boundary is trivial [KV83], [Ro81] (but there may also be measures with a non-trivial boundary; see [KV83], [Ka85a] for examples).

If G is virtually nilpotent (in particular, abelian), then (Γ,ν) is always trivial [DM61]. For abelian groups this result is usually referred to as the Choquet–Deny theorem. In fact, Choquet and Deny proved a stronger result: all minimal harmonic functions on such groups are multiplicative characters [CD60] (cf. below **1.3.5**), whereas triviality of the Poisson boundary for abelian groups had been earlier proved by Blackwell [Bl55].

1.2.9. Example. Let G be a free group with generators a, b. Consider the measure $\mu(a)=\mu(b)=1/2$. This is the simplest example of a random walk with a non-trivial Poisson boundary. Indeed, one can easily see that two paths from the path space $(G^{\mathbb{Z}_+},\mathbf{P})$ are $\sim$ equivalent iff they coincide. Thus, the Poisson partition η coincides with the point partition of the path space, and the Poisson boundary (Γ,ν) is the set of all infinite words in the alphabet a, b with the Bernoulli measure with the weights $(1/2, 1/2)$ on it. More generally, if $\mathrm{supp}\,\mu$ generates a free subsemigroup of G, then the Poisson boundary is the set of infinite words in the alphabet $\mathrm{supp}\,\mu$ with the Bernoulli measure $\mu\otimes\mu\otimes\cdots$. Obviously, in this situation the harmonic measure ν is *not* quasi-invariant with respect to the action of G.

1.3. Bounded harmonic functions and the Poisson formula.

1.3.1. A function f is called *μ-harmonic* if $Pf = f$, where $P = P_\mu$ is the Markov operator of the random walk (G,μ) introduced in **1.1.2**. Denote by $H^\infty(G,\mu)$ the Banach space of bounded μ-harmonic functions on G with the sup-norm.

Theorem. *The formulas*

$$F(\mathbf{bnd}\,\boldsymbol{x}) = \lim_{n\to\infty} f(x_n)\,, \qquad f(g) = \langle F, g\nu\rangle\,, \quad g\in G$$

state an isometric isomorphism between the spaces $H^\infty(G,\mu)$ and $L^\infty(\Gamma,[\nu_m])$.

Proof. Denote by $\mathcal{A}_0^n$ the σ-algebra in the path space $G^{\mathbb{Z}_+}$ generated by the positions of the random walk at times $0,1,\ldots,n$. Then a function f on G is μ-harmonic if and only if the sequence of functions $\varphi_n(\boldsymbol{x}) = f(x_n)$ on the path space is a *martingale* with respect to the increasing sequence of σ-algebras $\mathcal{A}_0^n$, because the martingale condition $\mathbf{E}(\varphi_{n+1}|\mathcal{A}_0^n) = \varphi_n$ is precisely the harmonicity condition. Thus, by the Martingale

Convergence Theorem for $\mathbf{P}_m$-a.e. sample path $\boldsymbol{x} = \{x_n\}$ there exists a limit $\lim f(x_n)$, which is obviously measurable with respect to the σ-algebra $\mathcal{A}_T$. Since the Poisson boundary Γ is the quotient of the path space determined by the σ-algebra $\mathcal{A}_T$, it means that there is a function $F \in L^\infty(\Gamma, \nu)$ such that $\lim f(x_n) = F(\mathbf{bnd}\, \boldsymbol{x})$.

Conversely, let $F \in L^\infty(\Gamma, [\nu_m])$. Since the measure ν is μ-stationary,

$$Pf(g) = \sum_h \mu(h) f(gh) = \sum_h \mu(h)\langle F, gh\nu\rangle = \langle F, g\mu\nu\rangle = \langle F, g\nu\rangle = f(g) ,$$

i.e., the function f is μ-harmonic. It remains to check that $\lim f(x_n) = F(\mathbf{bnd}\, \boldsymbol{x})$, where it is sufficient to consider just an indicator function $F = \mathbf{1}_A$, $A \subset \Gamma$. By the definition of the harmonic measure,

$$f(g) = \langle F, g\nu\rangle = \langle F, \nu_g\rangle = \mathbf{P}_g(\mathbf{bnd}^{-1} A) .$$

Moreover, since the set $\mathbf{bnd}^{-1} A$ is T-invariant, by the Markov property

$$f(g) = \mathbf{P}_g(\mathbf{bnd}^{-1} A) = \mathbf{P}_\theta(\mathbf{bnd}^{-1} A | x_n = g) \tag{1.3.1}$$

for any $n \geq 0$ and any probability measure θ on G with $\operatorname{supp} \theta = G$. Again by the Markov property, the event $\mathbf{bnd}^{-1} A$ is conditionally independent of the σ-algebra $\mathcal{A}_0^{n-1}$ under the condition $[x_n = g]$, so that (1.3.1) can be rewritten as

$$f(x_n) = \mathbf{P}_\theta(\mathbf{bnd}^{-1} A | \mathcal{A}_0^n) .$$

Hence a.e.

$$f(x_n) \to \mathbf{1}_{\mathbf{bnd}^{-1} A}(\boldsymbol{x}) = \mathbf{1}_A(\mathbf{bnd}\, \boldsymbol{x}) = F(x) ,$$

because the limit of the increasing sequence of σ-algebras $\mathcal{A}_0^n$ is the full σ-algebra of the path space. □

Since any G-invariant harmonic function on G is obviously constant, we obtain

Corollary. *The action of the group G on the Poisson boundary Γ is ergodic with respect to the harmonic measure type $[\nu_m]$.*

1.3.2.

> *Below we always consider the Poisson boundary Γ of the couple (G, μ) as a measure space with the harmonic measure $\nu = \nu_e$ determined by the group identity e as a starting point. Unless otherwise specified, no conditions are imposed neither on the group* $\operatorname{gr}(\mu)$ *nor on the semigroup* $\operatorname{sgr}(\mu)$ *generated by the support of the measure μ.*

In this situation Theorem 1.3.1 yields an isometric isomorphism between the space of bounded μ-harmonic functions on $\operatorname{sgr}(\mu)$ and the space $L^\infty(\Gamma, \nu)$. Since $g\nu \prec \nu$ for any $g \in \operatorname{sgr}(\mu)$ (see Proposition 1.2.5), the Poisson formula can be then rewritten using the *Poisson kernel* $\Pi(g,\gamma) = dg\nu/d\nu(\gamma)$ as

$$f(g) = \langle F, g\nu \rangle = \int F(\gamma)\Pi(g,\gamma)\, d\nu(\gamma) \;.$$

In other words,

$$f = \int F(\gamma)\varphi_\gamma \, d\nu(\gamma) = \int \varphi_\gamma \, d\nu_f(\gamma) \;, \tag{1.3.2}$$

where $\varphi_\gamma = \Pi(\cdot,\gamma)$ are μ-harmonic functions on $\operatorname{sgr}(\mu)$ given by Radon–Nikodym derivatives of the translations of the measure ν, and $\nu_f = F\nu$ is the *representing measure* of f.

1.3.3. Denote by $H_1^+(G,\mu)$ the convex set of all non-negative harmonic functions on $\operatorname{sgr}(\mu)$ normalized by the condition $f(e) = 1$. Any function $f \in H_1^+(G,\mu)$ determines a new Markov chain (the *Doob transform*) on $\operatorname{sgr}(\mu)$ whose transition probabilities

$$p^f(x,y) = \mu(x^{-1}y)\frac{f(y)}{f(x)} \;,$$

are "cohomologous" to the transition probabilities (1.1.1) of the original random walk. For any cylinder subset (1.1.4) of the path space the Markov measure $\mathbf{P}^f$ in the space of sample paths of the Doob transform (with the initial distribution δ_e) is connected with the measure $\mathbf{P}$ by the formula

$$\mathbf{P}^f(C_{e,g_1,\dots,g_n}) = \mathbf{P}(C_{e,g_1,\dots,g_n})f(g_n) \;, \tag{1.3.3}$$

i.e., the map (1.3.3) is a convex embedding of $H_1^+(G,\mu)$ into the space of Markov measures on $G^{\mathbb{Z}_+}$.

1.3.4. If A is a measurable subset of the Poisson boundary with $\nu(A) > 0$, then by the Markov property for any cylinder set $C_{e,g_1,\dots,g_n}$

$$\mathbf{P}(C_{e,g_1,\dots,g_n} \cap \mathbf{bnd}^{-1}A) = \mathbf{P}(C_{e,g_1,\dots,g_n})\mathbf{P}_{g_n}(\mathbf{bnd}^{-1}A) = \mathbf{P}(C_{e,g_1,\dots,g_n})g_n\nu(A) \;,$$

whence

$$\mathbf{P}(C_{e,g_1,\dots,g_n}|\mathbf{bnd}^{-1}A) = \frac{\mathbf{P}(C_{e,g_1,\dots,g_n})g_n\nu(A)}{\mathbf{P}(\mathbf{bnd}^{-1}A)} = \mathbf{P}(C_{e,g_1,\dots,g_n})\frac{g_n\nu(A)}{\nu(A)} \;,$$

i.e., the conditional measure $\mathbf{P}^A(\cdot) = \mathbf{P}(\cdot|\mathbf{bnd}^{-1}A)$ is the Doob transform of the measure $\mathbf{P}$ determined by the normalized harmonic function $\varphi_A(x) = x\nu(A)/\nu(A)$. Now,

$$\varphi_A = \frac{1}{\nu(A)}\int_A \varphi_\gamma \, d\nu(\gamma) \;, \qquad \varphi_\gamma(x) = dx\nu/d\nu(\gamma) \;,$$

cf. (1.3.2), whence by the convexity of the Doob transform

$$\mathbf{P}^A = \frac{1}{\nu(A)}\int_A \mathbf{P}^\gamma \, d\nu(\gamma) \;,$$

where $\mathbf{P}^\gamma$ are Doob transforms determined by the functions φ_γ, which yields

Theorem. *The measures*

$$\mathbf{P}^\gamma(C_{e,g_1,\dots,g_n}) = \mathbf{P}(C_{e,g_1,\dots,g_n}|\gamma) = \mathbf{P}(C_{e,g_1,\dots,g_n})\frac{dg_n\nu}{d\nu}(\gamma)$$

corresponding to the Markov operators P_γ on $\operatorname{sgr}(\mu)$ *with transition probabilities*

$$p_\gamma(x,y) = \mu(x^{-1}y)\frac{dy\nu}{dx\nu}(\gamma)$$

are the canonical system of conditional measures of the measure $\mathbf{P}$ *with respect to the Poisson boundary.*

Corollary 1. *The Radon–Nikodym derivatives $\varphi_\gamma(x) = dx\nu/d\nu(\gamma),\ x \in \operatorname{sgr}(\mu), \gamma \in \Gamma$ separate points of the space (Γ,ν).*

Proof. Since the conditional measures in the path space corresponding to different points $\gamma \in \Gamma$ are pairwise singular, different points $\gamma \in \Gamma$ determine different functions φ_γ. □

Corollary 2. *The harmonic functions $\varphi_\gamma(x) = dx\nu/d\nu(\gamma)$ are a.e. minimal, i.e., can not be decomposed into a non-trivial linear combination of positive harmonic functions.*

Proof. The measures $\mathbf{P}^\gamma = \mathbf{P}(\cdot|\gamma)$ are conditional measures on *ergodic components* of the time shift, so that they are ergodic themselves. By convexity of the Doob transform (1.3.3) it implies minimality of φ_γ. □

1.3.5. The *Martin boundary* of the random walk (G,μ) is the boundary of the *Martin compactification* of the group G determined by the measure μ. It admits a realization as a subset of the space of positive functions on G normalized by the condition $\varphi(e) = 1$ and endowed with the topology of pointwise convergence. The Martin boundary contains all minimal μ-harmonic functions, and any positive μ-harmonic function f can be decomposed as an integral $f = \int \varphi\, d\nu_f(\varphi)$ of *minimal* (extremal) harmonic functions with respect to a uniquely determined *representing measure* ν_f [DY69], [Br71]. Because of the uniqueness of the representing measure, for bounded harmonic functions this decomposition coincides with the one given by the Poisson formula (1.3.2), see Corollary 2 of Theorem 1.3.4.

In particular, the Martin boundary considered as a measure space with the representing measure of the constant function $\mathbf{1}$ coincides as a measure G-space with the Poisson boundary (Γ,ν). However, we emphasize that the natures of the Poisson and the Martin boundaries are different: the former is a measure space, whereas the latter is a topological space.

1.4. Quotients of the Poisson boundary (μ-boundaries).

1.4.1. Definition. The quotient (Γ_ξ, ν_ξ) of the Poisson boundary (Γ, ν) with respect to a certain G-invariant measurable partition ξ is called a *μ-boundary.*

Another way of defining a μ-boundary is to say that it is a G-space with a μ-stationary measure λ such that $x_n\lambda$ weakly converges to a δ-measure for $\mathbf{P}$-a.e. path $\{x_n\}$ of the random walk (G, μ) [Fu73].

The Poisson boundary itself is the maximal μ-boundary, and the singleton is the minimal μ-boundary. We shall denote by $\mathbf{bnd}_\xi$ the canonical projection

$$\mathbf{bnd}_\xi : (G^{\mathbb{Z}_+}, \mathbf{P}) \to (\Gamma, \nu) \to (\Gamma_\xi, \nu_\xi) ,$$

and by η_ξ the corresponding partition of the path space (recall that the partition of the path space corresponding to the Poisson boundary is denoted η). The measure ν_ξ and almost all conditional measures on the fibers of the projection $\Gamma \to \Gamma_\xi$ are purely non-atomic (unless $\Gamma_\xi = \{\cdot\}$ or $\Gamma_\xi = \Gamma$, respectively) [Ka95].

1.4.2. Any G-space which is a $\sim$-measurable image of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ is a μ-boundary (recall that $\sim$ is the orbit equivalence relation (1.2.2) of the time shift T). In other words, if π is a T-invariant equivariant measurable map from the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ to a G-space B, then $(B, \pi(\mathbf{P}))$ is a μ-boundary. For example, such a map arises in the situation when G is embedded into a topological G-space X, and $\mathbf{P}$-a.e. sample path $\boldsymbol{x} = \{x_n\}$ converges to a limit $x_\infty = \pi(\boldsymbol{x}) \in X$. In this situation we shall say that the limit measure $\pi(\mathbf{P})$ is the *harmonic measure* of the random walk (G, μ) with respect to the embedding $G \hookrightarrow X$.

Another example of a μ-boundary arises from taking a quotient of the group G by a normal subgroup $H \subset G$. Denote by μ' the image of the measure μ on the quotient group $G' = G/H$. Then the Poisson boundary (Γ', ν') of the random walk (G', μ') is the space of ergodic components of the Poisson boundary (Γ, ν) of the random walk (G, μ) with respect to the action of H [Ka95]. In other words, (Γ', ν') is the quotient of (Γ, ν) with respect to the G-invariant measurable partition generated by the action of H.

1.4.3. For any group G with a probability measure μ on it let $\widetilde{G}$ be the free group with the set of free generators $A \cong \operatorname{supp} \mu$ (we assume that inverse elements from $\operatorname{supp} \mu$ are independent generators of $\widetilde{G}$, so that the subsemigroup generated by A is free) and with the measure $\widetilde{\mu} \cong \mu$ on it. The Poisson boundary of $(\widetilde{G}, \widetilde{\mu})$ is the set (A^∞, μ^∞) of infinite words in the alphabet A with the Bernoulli measure (see Example 1.2.9), so that the Poisson boundary of (G, μ) is the quotient of (A^∞, μ^∞) with respect to the action of the kernel H of the projection $\widetilde{G} \to G$. This action consists in applying to infinite words from A^∞ all possible relations between generators from A present in the group G. However, contrary to a suggestion formulated in the pioneering paper [DM61], obtaining an effective description of the Poisson boundary of (G, μ) in this way turns out to be quite a hard task. Already in the case of an *arbitrary* (not necessarily concentrated on the set of generators) measure μ on a free semigroup it is unknown whether the set of infinite words (which is obviously a μ-boundary with the natural measure obtained by taking infinite products of μ-distributed increments) is indeed the whole Poisson boundary. Moreover, there are groups (see examples from [KV83], [Ka85a]) for which a description of the Poisson boundary in terms of any kind of infinite words seems quite unlikely at all.

1.4.4. Generally speaking, the problem of describing the Poisson boundary of (G, μ) consists of the following two parts:

(1) To find (in geometric or combinatorial terms) a μ-boundary (B, λ) which is *a priori* just the quotient (Γ_ξ, ν_ξ) of the Poisson boundary with respect to a certain G-invariant partition ξ;
(2) To show that this μ-boundary is maximal, i.e., that ξ is in fact the point partition of the Poisson boundary.

These two parts are quite different. First one has to exhibit a certain system of invariants ("patterns") of the behaviour of the random walk at infinity, and then to show completeness of this system, i.e., that these patterns completely describe the behaviour at infinity. A particular case of the problem of describing the Poisson boundary is proving its triviality.

1.4.5. Definition. A compactification of the group G is called *μ-maximal* if sample paths of the random walk (G, μ) converge a.e. in this compactification (so that it is a μ-boundary), and this μ-boundary is in fact isomorphic to the Poisson boundary of (G, μ).

This property means that the compactification is indeed *maximal in a measure theoretical sense*, i.e., there is no way (up to measure 0) of splitting further the boundary points of this compactification. We shall give in Section 2 general geometric criteria for maximality of μ-boundaries and μ-maximality of group compactifications using a quantitative approach based on the entropy theory of random walks.

1.4.6. Let now (Γ_ξ, ν_ξ) be a μ-boundary. Then for ν_ξ-a.e. $\gamma_\xi \in \Gamma_\xi$

$$\frac{dg\nu_\xi}{d\nu_\xi}(\gamma_\xi) = \int \frac{dg\nu}{d\nu}(\gamma)\, d\nu(\gamma|\gamma_\xi) ,$$

where $\nu(\cdot|\gamma_\xi)$ are the conditional measures of the measure ν on the fibers of the projection $\Gamma \to \Gamma_\xi$, $\gamma \mapsto \gamma_\xi$. Then Theorem 1.3.4 and convexity of the Doob transform (1.3.3) imply

Theorem. *The conditional measures of the measure* $\mathbf{P}$ *with respect to a μ-boundary* (Γ_ξ, ν_ξ) *are*

$$\mathbf{P}^{\gamma_\xi}(C_{e,g_1,\dots,g_n}) = \mathbf{P}(C_{e,g_1,\dots,g_n}|\gamma_\xi) = \mathbf{P}(C_{e,g_1,\dots,g_n})\frac{dg_n\nu_\xi}{d\nu_\xi}(\gamma_\xi) , \qquad \gamma_\xi \in \Gamma_\xi$$

and correspond to the Markov operators P_{γ_ξ} *on* $\operatorname{sgr}(\mu)$ *with transition probabilities*

$$p^{\gamma_\xi}(x, y) = \mu(x^{-1}y)\frac{dy\nu_\xi}{dx\nu_\xi}(\gamma_\xi) .$$

Another proof could be also obtained by directly reproducing the argument from the proof of Theorem 1.3.4.

Corollary. *The Radon–Nikodym derivatives $dx\nu_\xi/d\nu_\xi(\gamma_\xi)$, $x \in \operatorname{sgr}(\mu), \gamma_\xi \in \Gamma_\xi$ separate points of the space (Γ_ξ, ν_ξ).*

1.5. The tail boundary.

1.5.1. Another measure-theoretic boundary associated with a Markov operator is the *tail boundary*. Its definition is analogous to the definition of the Poisson boundary, with the grand orbit equivalence relation $\sim$ (1.2.2) being replaced with the small (or, synchronous) orbit equivalence relation $\approx$:

$$\boldsymbol{x} \approx \boldsymbol{x}' \iff \exists\, n \geq 0 : T^n \boldsymbol{x} = T^n \boldsymbol{x}' \,.$$

An important difference (crucial for what follows) is that unlike the σ-algebra $\mathcal{A}_T$ from the definition of the Poisson boundary, the *tail σ-algebra* $\mathcal{A}^\infty$ of all measurable unions of $\approx$-classes can be presented in a canonical way as the limit of a decreasing sequence of σ-algebras. Namely, $\mathcal{A}^\infty$ is the limit of the σ-algebras $\mathcal{A}_n^\infty$ determined by the positions of sample paths at times $\geq n$. One can say that the tail boundary completely describes the *stochastically significant behaviour* of the Markov chain at infinity.

In the language of the corresponding measurable partitions of the path space, the *tail partition* α^∞ (which is the measurable envelope of the equivalence relation $\approx$) is the measurable intersection $\bigwedge \alpha_n^\infty$ of the decreasing sequence of measurable partitions α_n^∞ corresponding to σ-algebras $\mathcal{A}_n^\infty$ (i.e., two paths $\boldsymbol{x}$ and $\boldsymbol{x}'$ belong to the same class of the partition α_n^∞ iff $x_i = x_i'$ for all $i \geq n$).

1.5.2. The tail boundary is the Poisson boundary for the *space-time* operator $\widetilde{P}f(\cdot, n) = Pf(\cdot, n+1)$ on $X \times \mathbb{Z}$, so that it gives integral representation of *bounded harmonic sequences* $f_n = Pf_{n+1}$ on X (which are counterparts of so-called *parabolic harmonic functions* in the classical setup). The tail boundary is endowed with a natural action of the time shift T induced by the time shift in the path space, and the Poisson boundary is the space of ergodic components of the tail boundary with respect to T. Triviality of the tail boundary means that the Markov operator P is *mixing* in the same way as triviality of the Poisson boundary is equivalent to ergodicity of P (e.g., see [De76], [Ro81]).

1.5.3. The Poisson and the tail boundaries are sometimes confused, and, indeed, they do coincide for "most common" Markov operators (such operators are called *steady* in [Ka92]). General criteria of triviality of these boundaries and of their coincidence for an arbitrary Markov operator are provided by *0–2 laws*, see [De76], [Ka92]. In particular,

Theorem [Ka92]. *The tail and the Poisson boundaries coincide $\mathbf{P}_\theta$ – mod 0 for a given initial distribution θ on G iff for any integers $k, d \geq 0$ and any probability measure $\lambda \prec \theta\mu_k \wedge \theta\mu_{k+d}$*

$$\lim_{n\to\infty} \|\lambda\mu_n - \lambda\mu_{n+d}\| = 0 \,.$$

Otherwise there exists $d > 0$ with the property that for every $\varepsilon > 0$ there are $k \geq 0$ and a probability measure $\lambda \prec \theta\mu_k \wedge \theta\mu_{k+d}$ such that

$$\lim_{n\to\infty} \|\lambda\mu_n - \lambda\mu_{n+d}\| > 2 - \varepsilon \,.$$

1.5.4. If for certain $k,d\ge 0$ the measures μ_k and μ_{k+d} are non-singular, then for any probability measure λ on G

$$\lim_n \|\lambda\mu_n-\lambda\mu_{n+d}\| = \lim_n \|\lambda\mu_n(\mu_k-\lambda\mu_{k+d})\| \le \|\mu_k-\mu_{k+d}\| < 2 \ ,$$

so that the second part of Theorem 1.5.3 applied to the initial distribution $\theta=\delta_e$ implies

Theorem [De80], [KV83]. *The Poisson and the tail boundaries coincide* $\mathbf{P}-$ mod 0.

Note that the first part of Theorem 1.5.3 then implies that $\lim\|\mu_n-\mu_{n+d}\|=0$ whenever the measures μ_k and μ_{k+d} are non-singular for a certain k (cf. [Fo75]).

1.5.5. Coincidence of the Poisson and the tail boundary with respect to a single point initial distribution for random walks on groups is a key ingredient of the entropy theory of random walks (see Section 1.6). As Theorem 1.5.3 shows, the reason for their discrepancy for a non-trivial initial distribution θ is existence of such $d>0$ that for any $n>0$ the convolutions μ_n and μ_{n+d} are pairwise singular. The minimal D with the property that $\|\mu_n-\mu_{n+D}\|\to 0$ is called the *period* of the measure μ. If $D<\infty$, then one can easily see that the tail boundary for an arbitrary initial distribution θ is a $\mathbb{Z}_D$-cover over the Poisson boundary. In the case $D=\infty$ a similar description was obtained in a recent paper [Ja95].

1.6. Entropy and triviality of the Poisson boundary.

1.6.1. *From now on we shall assume that the measure μ has finite entropy*

$$H(\mu)=\sum_{g\in G} -\mu(g)\log\mu(g)\ .$$

Lemma. *The sequence $H(\mu_n)$ of entropies of n-fold convolutions of the measure μ is subadditive.*

Proof. The measure μ_{n+m} is the image of the product measure $\mu_n\otimes\mu_m$ under the map $(g_1,g_2)\mapsto g_1g_2$, whence by the well known properties of the entropy (e.g., see [Ro67]) $H(\mu_n)+H(\mu_m)=H(\mu_n\otimes\mu_m)\ge H(\mu_{n+m})$. □

1.6.2. Definition [Av72]. The limit (which exists by Lemma 1.6.1)

$$h(G,\mu)=\lim_{n\to\infty}\frac{H(\mu_n)}{n}$$

is called the *entropy of the random walk* (G,μ).

1.6.3. Definition. A probability measure Λ on $G^{\mathbb{Z}_+}$ has *asymptotic entropy* $\mathbf{h}(\Lambda)$ if it has the following Shannon–Breiman–McMillan type *equidistribution property*:

$$-\frac{1}{n}\log\Lambda(C^n_{x_n})\to\mathbf{h}(\Lambda)$$

for Λ-a.e. $\boldsymbol{x}=\{x_n\}\in G^{\mathbb{Z}_+}$ and in the space $L^1(\Lambda)$.

Note that if λ_n is the one-dimensional distribution of the measure Λ at time n, then

$$-\int\log\Lambda(C^n_{x_n})\,d\Lambda(\boldsymbol{x})=-\sum_{x_n}\log\lambda_n(x_n)\lambda_n(x_n)=H(\lambda_n)\,, \tag{1.6.1}$$

so that L^1-convergence in the above definition implies that $H(\lambda_n)/n\to\mathbf{h}(\Lambda)$.

1.6.4. Theorem [De80], [KV83]**.** *The asymptotic entropy* $\mathbf{h}(\mathbf{P})$ *of the measure* $\mathbf{P}$ *exists, and* $\mathbf{h}(\mathbf{P})=h(G,\mu)$.

Proof. Consider the functions $f_n(\boldsymbol{x})=-\log\mu_n(x_n)$ on the path space. By formula (1.6.1) and Lemma 1.6.1 they are integrable. Since

$$\mu_{n+m}(x_{n+m})=p_{n+m}(e,x_{n+m})\ge p_n(e,x_n)p_m(x_n,x_{n+m})=\mu_n(x_n)\mu_m(x_n^{-1}x_{n+m})\,,$$

we have the *subadditivity* property

$$f_{n+m}(\boldsymbol{x})\le f_n(\boldsymbol{x})+f_m(U^n\boldsymbol{x})\,,$$

where U is the measure preserving transformation of $(G^{\mathbb{Z}_+},\mathbf{P})$ introduced in **1.2.6**, so that the claim at once follows from Kingman's Subadditive Ergodic Theorem. □

1.6.5. Recall that the *entropy* $H(\xi)=H(X,m,\xi)$ *of a countable measurable partition* ξ of a Lebesgue space (X,m) is defined as the entropy of the quotient probability distribution m_ξ on the quotient space X_ξ. It can be written down as

$$H(\xi)=H(m_\xi)=-\int\log m(\xi_x)\,dm(x)\,,$$

where $\xi_x\subset X$ denotes the element of the partition ξ containing the point x. If ζ is another measurable partition of the same space (X,m), then the (mean) *conditional entropy* of ξ with respect to ζ is defined as

$$H(\xi|\zeta)=\int H\big(X,m(\cdot|x_\zeta),\xi\big)\,dm_\zeta(x_\zeta)=-\int\log m(\xi_x|x_\zeta)\,dm(x)\,,$$

where $x\mapsto x_\zeta$ is the canonical projection $(X,m)\to(X_\zeta,m_\zeta)$, and $m(\cdot|x_\zeta)$ are the conditional measures of m on the fibers of this canonical projection. In other words, $H(\xi|\zeta)$ is the average of entropies of ξ with respect to conditional measures of the partition ζ.

We shall need the following properties of the conditional entropy (see [Ro67]):

(i) If η is a refinement of ζ (notation: $\zeta \preccurlyeq \eta$), then $H(\xi|\zeta) \ge H(\xi|\eta)$ with the equality iff m-a.e. $m(\xi_x|x_\zeta) = m(\xi_x|x_\eta)$. In particular, comparing ζ with the point partition and with the trivial partition of the space X, we get the inequality $0 \le H(\xi|\zeta) \le H(\xi)$; the equality in the left-hand side holds iff $\xi \preccurlyeq \zeta$, and in the right-hand side iff ξ and ζ are independent.

(ii) If $\zeta_n \downarrow \zeta$ (i.e., $\zeta_{n+1} \preccurlyeq \zeta_n$ for any n, and ζ is the maximal measurable partition such that $\zeta \preccurlyeq \zeta_n$ for all n), then $H(\xi|\zeta_n) \uparrow H(\xi|\zeta)$.

1.6.6. Denote by α_1^k the partition of the path space (X, m) determined by the positions of the random walk at times $1, 2, \dots, k$, i.e., two sample paths $\boldsymbol{x}, \boldsymbol{x}'$ belong to the same class of α_1^k iff $x_i = x_i'$ for all $i = 1, 2, \dots, k$. The quotient of the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ determined by the partition α_1^k is the space of initial segments (up to time k) of sample paths, and it is isomorphic to the space of first k increments of the random walk. Let $\alpha = \alpha_1^1$. Since the increments are independent and all have distribution μ, we obtain that $H(\alpha_1^k) = kH(\mu) = kH(\alpha)$.

Lemma. *The conditional entropy of a partition $\alpha_1^k, k \ge 1$ with respect to the Poisson partition η is*

$$H(\alpha_1^k|\eta) = kH(\alpha|\eta) = k\big[H(\mu) - h(G,\mu)\big] .$$

Proof. We shall use the fact that η is the decreasing limit of the coordinate partitions α_n^∞ (see **1.5.1** and Theorem 1.5.4). By the Markov property for a given sample path $\boldsymbol{x} = \{x_n\} \in G^{\mathbb{Z}_+}$

$$\begin{aligned}
\mathbf{P}((\alpha_1^k)_{\boldsymbol{x}}|\boldsymbol{x}_{\alpha_n^\infty}) = \mathbf{P}(C_{e,x_1,\dots,x_k}|C^n_{x_n}) &= \frac{\mathbf{P}(C_{e,x_1,\dots,x_k} \cap C^n_{x_n})}{\mathbf{P}(C^n_{x_n})} \\
&= \frac{\mu(x_1)\mu(x_1^{-1}x_2)\dots\mu(x_{k-1}^{-1}x_k)\mu_{n-k}(x_k^{-1}x_n)}{\mu_n(x_n)} \\
&= \frac{\mu(h_1)\dots\mu(h_k)\mu_{n-k}(h_{k+1}\dots h_n)}{\mu_n(h_1\dots h_n)} ,
\end{aligned}$$

where h_i are the independent μ-distributed increments (1.1.2) of the random walk, whence

$$H(\alpha_1^k|\alpha_n^\infty) = kH(\mu) + H(\mu_{n-k}) - H(\mu_n)$$

(we are assuming that $k \le n$). Now, by property 1.6.5 (ii)

$$H(\alpha_1^k|\eta) = \lim_n H(\alpha_1^k|\alpha_n^\infty) = kH(\mu) - \lim_n\big[H(\mu_n) - H(\mu_{n-k})\big] .$$

By Definition 1.6.2, once the limit in the right-hand side exists, it must be equal to $kh(G,\mu)$. □

1.6.7. Theorem [De80], [KV83]. *If the entropy $H(\mu)$ of the measure μ is finite, then the Poisson boundary of the random walk (G,μ) is trivial* $\mathbf{P}-\mathrm{mod}\ 0$ *iff* $h(G,\mu)=0$.

Proof. If $h(G,\mu)=0$, then by Lemma 1.6.6 and property 1.6.5 (i) the Poisson partition η is independent of all coordinate partitions α_1^k, which by the Kolmogorov 0–1 Law is only possible if η is trivial. Conversely, if η is trivial, then $H(\alpha_1^k|\eta)=H(\alpha_1^k)=kH(\mu)$, whence $h(G,\mu)=0$. □

Theorem 1.6.4 now implies

Corollary. *The Poisson boundary is trivial iff there exist $\varepsilon>0$ and a sequence of sets A_n such that $\mu_n(A_n)>\varepsilon$ and $\log|A_n|=o(n)$.*

1.7. Entropy of conditional walks and maximality of μ-boundaries.

1.7.1. Let now ξ be a G-invariant partition of the Poisson boundary, and (Γ_ξ,ν_ξ) – the corresponding μ-boundary.

Lemma. *For any $k\ge 1$*

$$H(\alpha_1^k|\eta_\xi)=kH(\alpha|\eta_\xi)=k\left[H(\mu)-\int\log\frac{dx_1\nu_\xi}{d\nu_\xi}(\mathbf{bnd}_\xi\,\boldsymbol{x})\,d\mathbf{P}(\boldsymbol{x})\right] .$$

Proof. Given a path $\boldsymbol{x}=\{x_n\}\in G^{\mathbb{Z}_+}$, the element $(\alpha_1^k)_{\boldsymbol{x}}$ of the partition α_1^k containing $\boldsymbol{x}$ is the cylinder $C_{e,x_1,\dots,x_k}$, and the image $\boldsymbol{x}_{\eta_\xi}$ of $\boldsymbol{x}$ in Γ_ξ is $\mathbf{bnd}_\xi\,\boldsymbol{x}$, whence by Theorem 1.4.6 the corresponding conditional probability is

$$\mathbf{P}\big((\alpha_1^k)_{\boldsymbol{x}}|\boldsymbol{x}_{\eta_\xi}\big)=\mathbf{P}(C_{e,x_1,\dots,x_k}|\mathbf{bnd}_\xi\,\boldsymbol{x})=\mathbf{P}(C_{e,x_1,\dots,x_k})\frac{dx_k\nu_\xi}{d\nu_\xi}(\mathbf{bnd}_\xi\,\boldsymbol{x}) ,$$

and

$$H(\alpha_1^k|\eta_\xi)=kH(\mu)-\int\log\frac{dx_k\nu_\xi}{d\nu_\xi}(\mathbf{bnd}_\xi\,\boldsymbol{x})\,d\mathbf{P}(\boldsymbol{x}) .$$

Now, telescoping

$$\begin{aligned}\frac{dx_k\nu_\xi}{d\nu_\xi}(\mathbf{bnd}_\xi\,\boldsymbol{x})&=\frac{dh_1\dots h_k\nu_\xi}{d\nu_\xi}(\mathbf{bnd}_\xi\,\boldsymbol{x})\\&=\prod_{i=1}^k\frac{dh_i\nu_\xi}{d\nu_\xi}(x_{i-1}^{-1}\mathbf{bnd}_\xi\,\boldsymbol{x})=\prod_{i=1}^k\frac{d(U^{i-1}\boldsymbol{x})_1\nu_\xi}{d\nu_\xi}(\mathbf{bnd}_\xi\,U^{i-1}\boldsymbol{x}) ,\end{aligned}\tag{1.7.1}$$

and using Lemma 1.2.6 we get the claim. □

1.7.2. In particular, Lemma 1.7.1 implies finiteness of the integral

$$E(\Gamma_\xi,\nu_\xi)=\int\log\frac{dx_1\nu_\xi}{d\nu_\xi}(\mathbf{bnd}_\xi\,\boldsymbol{x})\,d\mathbf{P}(\boldsymbol{x}) .\tag{1.7.2}$$

This integral can be also rewritten in the following way. If $\lambda \prec \lambda'$ are two probability measures on a same space X, then the *Kullback–Leibler deviation* of λ from λ' is defined as

$$I(\lambda|\lambda') = \int \log \frac{d\lambda'}{d\lambda}(x)\, d\lambda'(x) \; .$$

Although the Kullback–Leibler deviation is not symmetric, it is non-negative (if finite), and equals 0 iff $\lambda = \lambda'$ [Ce82].

Using the change of variables $\boldsymbol{x} \mapsto (g, \boldsymbol{x}')$, $g = x_1, \boldsymbol{x}' = U\boldsymbol{x}$ we get from (1.7.2)

$$\begin{aligned} E(\Gamma_\xi, \nu_\xi) &= \sum_g \mu(g) \int \log \frac{dg\nu_\xi}{d\nu_\xi}(g\,\mathbf{bnd}_\xi\, \boldsymbol{x}')\, d\mathbf{P}(\boldsymbol{x}') \\ &= \sum_g \mu(g) \int \log \frac{dg\nu_\xi}{d\nu_\xi}(g\gamma_\xi)\, d\nu_\xi(\gamma_\xi) \\ &= \sum_g \mu(g) I(g^{-1}\nu_\xi|\nu_\xi) = \sum_g \mu(g) I(\nu_\xi|g\nu_\xi) \; . \end{aligned}$$

Thus, by Theorem 1.7.1

$$H(\alpha|\eta_\xi) = H(\mu) - E(\Gamma_\xi, \nu_\xi) = H(\mu) - \sum_g \mu(g) I(\nu_\xi|g\nu_\xi) \; . \tag{1.7.3}$$

1.7.3. Comparing (1.7.3) with Lemma 1.6.6, we get

$$h(G, \mu) = E(\Gamma, \nu) = \sum_g \mu(g)\, I(\nu|g\nu) \; . \tag{1.7.4}$$

Thus, the entropy $h(G, \mu)$ (initially defined in terms of convolutions of the measure μ) coincides with the average Kullback–Leibler deviation from the harmonic measure ν on the Poisson boundary to its translations (which is defined entirely in "boundary terms"). This result was first announced in [Ka83] (see also [KV83]), and it is the key to our criterion of maximality of μ-boundaries (Theorem 1.7.6).

1.7.4. Theorem. *Let $\xi \preccurlyeq \xi'$ be two G-invariant measurable partitions of the Poisson boundary (Γ, ν). Then $H(\alpha|\eta_\xi) \geq H(\alpha|\eta_{\xi'})$, and the equality holds iff $\xi = \xi'$.*

Proof. Obviously, if $\xi \preccurlyeq \xi'$, then $\eta_\xi \preccurlyeq \eta'_\xi$, so that the inequality follows from property 1.6.5 (i) of the conditional entropy. If $H(\alpha|\eta_\xi) = H(\alpha|\eta_{\xi'})$, then by Lemma 1.7.1 $H(\alpha_1^k|\eta_\xi) = H(\alpha_1^k|\eta_{\xi'})$ for any $k \geq 1$. By property 1.6.5 (i) it implies that for ν-a.e. point $\gamma \in \Gamma$ the conditional measures $\mathbf{P}^{\gamma_\xi}$ and $\mathbf{P}^{\gamma_{\xi'}}$ coincide, which by the Corollary of Theorem 1.4.4 is only possible when $\xi = \xi'$. □

Applying this Theorem to the case when ξ' is the point partition of the Poisson boundary and using formulas (1.7.3) and (1.7.4) we get

Corollary. *A μ-boundary (Γ_ξ, ν_ξ) coincides with the Poisson boundary iff $E(\Gamma_\xi, \nu_\xi) = h(G, \mu)$.*

Remark. In view of formula (1.7.3) Theorem 1.7.4 is equivalent to saying that if $\xi \preccurlyeq \xi'$, then $E(\Gamma_\xi, \nu_\xi) \le E(\Gamma_\xi, \nu_{\xi'})$ with the equality iff $\xi = \xi'$. This property can be also obtained from monotonicity properties of the Kullback–Leibler deviation [Ce82] and it was already known to Furstenberg [Fu71]. In some special situations one was able to use directly this property for proving maximality of μ-boundaries [Fu71], [Gu80b]. However, only identification of $E(\Gamma, \nu)$ with $h(G, \mu)$ makes it really operational for proving maximality of μ-boundaries.

1.7.5. Theorem. *Let ξ be a measurable G-invariant partition of the Poisson boundary (G, ν). Then for ν_ξ-a.e. point $\gamma_\xi \in \Gamma_\xi$ the asymptotic entropy (in the sense of Definition 1.6.3) of the conditional measure $\mathbf{P}^{\gamma_\xi}$ exists and is equal*

$$\mathbf{h}(\mathbf{P}^{\gamma_\xi}) = h(G, \mu) - E(\Gamma_\xi, \nu_\xi) = H(\alpha|\eta_\xi) - H(\alpha|\eta) .$$

Proof. We have to check that for ν_ξ-a.e. point $\gamma_\xi \in \Gamma_\xi$

$$-\frac{1}{n} \log \mathbf{P}^{\gamma_\xi}(C^n_{x_n}) \to h(G, \mu) - E(\Gamma_\xi, \nu_\xi)$$

for $\mathbf{P}^{\gamma_\xi}$-a.e. sample path $\boldsymbol{x} = \{x_n\}$ and in the space $L^1(\mathbf{P}^{\gamma_\xi})$. Since the measures $\mathbf{P}^{\gamma_\xi}$ are conditional measures of the measure $\mathbf{P}$, it amounts to proving that

$$-\frac{1}{n} \log \mathbf{P}^{\mathbf{bnd}_\xi \boldsymbol{x}}(C^n_{x_n}) \to h(G, \mu) - E(\Gamma_\xi, \nu_\xi)$$

$\mathbf{P}$-a.e. and in the space $L^1(\mathbf{P})$. By Theorem 1.4.6

$$\mathbf{P}^{\mathbf{bnd}_\xi \boldsymbol{x}}(C^n_{x_n}) = \mathbf{P}(C^n_{x_n}) \frac{d x_n \nu_\xi}{d\nu_\xi}(\mathbf{bnd}_\xi \boldsymbol{x}) ,$$

whence using (1.7.1) and applying the Birkhoff Ergodic Theorem to the transformation U, we obtain that

$$-\frac{1}{n} \log \mathbf{P}^{\mathbf{bnd}_\xi \boldsymbol{x}}(C^n_{x_n}) \underset{n\to\infty}{\longrightarrow} \mathbf{h}(\mathbf{P}) - E(\Gamma_\xi, \nu_\xi) = h(G, \mu) - E(\Gamma_\xi, \nu_\xi) .$$

□

1.7.6. Now, combining Corollary of Theorem 1.7.4 with Theorem 1.7.5 we get the following generalization of Theorem 1.6.7

Theorem. *A μ-boundary $(B, \lambda) \cong (\Gamma_\xi, \nu_\xi)$ is the Poisson boundary iff the asymptotic entropy $\mathbf{h}(\mathbf{P}^{\gamma_\xi})$ of almost all conditional measures of the measure $\mathbf{P}$ with respect to Γ_ξ vanishes.*

Corollary. *A μ-boundary $(B, \lambda) = (\Gamma_\xi, \nu_\xi)$ is the Poisson boundary iff for ν_ξ-a.e. point $\gamma_\xi \in \Gamma_\xi$ there exist $\varepsilon > 0$ and a sequence of sets $A_n = A_n(\gamma_\xi) \subset G$ such that*
(i) $\log |A_n| = o(n)$;
(ii) $p_n^{\gamma_\xi}(A_n) > \varepsilon$ *for all sufficiently large n, where $p_n^{\gamma_\xi}(g) = \mathbf{P}^{\gamma_\xi}(C_g^n)$ are the one-dimensional distributions of the measures $\mathbf{P}^{\gamma_\xi}$.*

Remark. Actually Theorem 1.7.5 shows that it is sufficient to check the conditions of the above Corollary not for almost all $\gamma_\xi \in \Gamma_\xi$, but just for a certain subset of Γ_ξ of bounded away from zero measure ν_ξ.

2. Geometric criteria of boundary maximality

2.1. Group compactifications and μ-boundaries.

2.1.1. Let $\overline{G} = G \cup \partial G$ be a compactification of a countable group G which is compatible with the group structure on G in the sense that the action of G on itself by left translations extends to an action on $\overline{G}$ by homeomorphisms. We introduce the following conditions on $\overline{G}$:

(CP) If a sequence $g_n \in G$ converges to a point from ∂G in the compactification $\overline{G}$, then the sequence $g_n x$ converges to the same limit for any $x \in G$.

(CS) The boundary ∂G consists of at least 3 points, and there is a G-equivariant map S assigning to pairs of distinct points (b_1, b_2) from ∂G non-empty subsets (*"strips"*) $S(b_-, b_+) \subset G$ such that for any 3 pairwise distinct points $\overline{b}_i \in \partial G$, $i = 0, 1, 2$ there exist neighbourhoods $\overline{b}_0 \in \mathcal{O}_0 \subset \overline{G}$ and $\overline{b}_i \in \mathcal{O}_i \subset \partial G$, $i = 1, 2$ with the property that

$$S(b_1, b_2) \cap \mathcal{O}_0 = \varnothing \qquad \forall\, b_i \in \mathcal{O}_i,\ i = 1, 2\ .$$

Condition (CP) is called *projectivity* in [Wo93], whereas condition (CS) means that points from ∂G are *separated by the strips* $S(b_1, b_2)$. As we shall see below (Theorem 2.4.7), it is often convenient to take for $S(b_1, b_2)$ the union of all bi-infinite geodesics in G (provided with a Cayley graph structure) which have b_1, b_2 as their endpoints.

2.1.2. Lemma. *Let $\overline{G} = G \cap \partial G$ be a compactification satisfying conditions* (CP), (CS), *and $(g_n) \subset G$ – a sequence such that $g_n \to \overline{b} \in \partial G$. Then for any non-atomic probability measure λ on ∂G the translations $g_n\lambda$ converge to the point measure $\delta_{\overline{b}}$ in the weak* topology.*

Proof. If $g_n b \to \overline{b}$ for all $b \in \partial G$, there is nothing to prove. Otherwise, passing to a subsequence we may assume that there exists $b_1 \in \partial G$ such that $g_n b_1 \to \overline{b}_1 \neq \overline{b}$. We claim that then $g_n b \to \overline{b}$ for all $b \neq b_1$. Indeed, if not, then passing again to a subsequence we may assume that there is $b_2 \neq b_1$ such that $g_n b_2 \to \overline{b}_2 \neq \overline{b}$. Take a point $x \in S(b_1, b_2)$, then by condition (CS) the only possible limit points of the sequence $g_n x$ are $\overline{b}_1$ or $\overline{b}_2$, which contradicts condition (CP). Since the measure λ is non-atomic, the claim implies that $g_n\lambda \to \delta_{\overline{b}}$.

Thus, any sequence (g_n) with $g_n \to \overline{b}$ has a subsequence (g_{n_k}) with $g_{n_k}\lambda \to \delta_{\overline{b}}$, so that $g_n\lambda \to \delta_{\overline{b}}$. □

Corollary. *If a compactification $\overline{G} = G \cap \partial G$ satisfies conditions* (CP), (CS), *λ is a non-atomic probability measure on ∂G, and $g_n\lambda \to \theta$ weakly for a sequence $g_n \to \infty$, then the limit θ is a point measure δ_b, $b \in \partial G$, and $g_n \to b$.*

2.1.3. Definition. A subgroup $G' \subset G$ is called *elementary* with respect to a compactification $\overline{G} = G \cap \partial G$ if G' fixes a finite subset of ∂G.

2.1.4. Theorem. *Let $\overline{G} = G \cap \partial G$ be a separable compactification of a countable group G satisfying conditions* (CP), (CS), *μ – a probability measure on G such that the subgroup $\mathrm{gr}\,(\mu)$ generated by its support is non-elementary with respect to this compactification, and $\mathbf{P}$ – the corresponding probability measure in the path space of the random walk (G, μ). Then $\mathbf{P}$-a.e. sample path $\boldsymbol{x} = \{x_n\}$ converges to a limit $x_\infty = \mathbf{bnd}\,\boldsymbol{x} \in \partial G$. The harmonic measure $\lambda = \mathbf{bnd}\,(\mathbf{P})$ is purely non-atomic, the measure space $(\partial G, \lambda)$ is a μ-boundary, and λ is the unique μ-stationary probability measure on ∂G.*

Proof. By compactness of ∂G there exists a μ-stationary probability measure λ on ∂G (take for λ any weak limit point of the sequence of Cesaro averages $(\mu\theta + \mu_2\theta + \dots \mu_n\theta)/n$, where θ is a probability measure on ∂G). The measure λ is purely non-atomic. Indeed, let m be the maximal weight of its atoms, and $A_m \subset \partial G$ be the finite set of atoms of weight m. Since λ is μ-stationary, $\lambda(b) = \sum_g \mu(g)\lambda(g^{-1}b)$ for any $b \in A_m$, whence A_m is $\mathrm{sgr}\,(\mu)^{-1}$-invariant, which by finiteness of A_m implies that A_m is also $\mathrm{gr}\,(\mu)$-invariant, the latter being impossible because the group $\mathrm{gr}\,(\mu)$ is non-elementary.

The measure λ is μ-stationary, so that for any function $F \in C(\partial G)$ the Poisson integral $f(g) = \langle F, g\lambda \rangle$ is a bounded μ-harmonic function, and the sequence of functions $\varphi_n(\boldsymbol{x}) = f(x_n) = \langle F, x_n\lambda \rangle$ on the path space is a.e. convergent (see Theorem 1.3.1). The boundary ∂G is separable, hence taking F from a dense countable subset of $C(\partial G)$ we obtain that almost every sequence of measures $x_n\lambda$ converges weakly to a probability measure $\lambda(\boldsymbol{x})$ (cf. [Fu71], [Ma91]). Since $\mathrm{gr}\,(\mu)$ is non-elementary, a.e. sample path $x = \{x_n\}$ is unbounded as a subset of G. By Corollary of Lemma 2.1.2, it implies that a.e. $x_n \to x_\infty = \mathbf{bnd}\,(\boldsymbol{x}) \in \partial G$ and $\lambda(\boldsymbol{x}) = \delta_{x_\infty}$.

Let $\nu = \mathbf{bnd}\,(\mathbf{P})$ be the distribution of the limit points x_∞, so that $(\partial G, \nu)$ is a μ-boundary (see **1.4.2**). By μ-stationarity of the measure λ

$$\lambda = \mu_n\lambda = \sum_g \mu_n(g)\, g\lambda = \int x_n\lambda\, d\mathbf{P}(\boldsymbol{x}) \qquad \forall n \geq 0\,,$$

whence, passing to the limit on n, we get that

$$\lambda = \int \lambda(\boldsymbol{x})\, d\mathbf{P}(\boldsymbol{x}) = \int \delta_{x_\infty}\, d\mathbf{P}(\boldsymbol{x}) = \mathbf{bnd}\,(\mathbf{P}) = \nu\,.$$

□

Corollary 1. *If* $\mathrm{gr}(\mu)$ *is non-elementary, then the Poisson boundary* $\Gamma(G,\mu)$ *is non-trivial.*

In particular, if $G' \subset G$ is non-elementary, then $\Gamma(G',\mu)$ is non-trivial for any μ with $\operatorname{supp}\mu = G'$, so that by [KV83]

Corollary 2. *Any non-elementary subgroup* $G' \subset G$ *is non-amenable.*

The latter corollary can be also directly deduced from the absence of G'-invariant probability measures on ∂G.

2.1.5. The idea of using the Martingale Convergence Theorem in combination with "contractivity" of the G-action on ∂G (all continuous measures on ∂G are contracted to points by converging sequences in G) for proving convergence of random walks goes back to Furstenberg [Fu71]. In the terminology of [Fu73] (see also [Ma91]) Theorem 2.1.4 implies that the G-action on ∂G is *mean proximal*. Here we use just the separation property (CS) (Lemma 2.1.2), whereas the standard approach consists in deducing mean proximality from proximality of G-action on the boundary with some additional contractivity conditions (cf. [Fu73], [Ma91, Proposition VI.2.13], [GR85], [CS89], [Wo89], [Wo93], [KM96]).

2.2. Gauges in groups.

2.2.1. Definition. An increasing sequence $\mathcal{G} = (\mathcal{G}_k)_{k\ge 1}$ of sets exhausting a countable group G is called a *gauge* on G. By

$$|g| = |g|_{\mathcal{G}} = \min\{k : g \in \mathcal{G}_k\}$$

we denote the corresponding *gauge function*.

We shall say that a gauge $\mathcal{G}$ is

- *symmetric* if all gauge sets $\mathcal{G}_k$ are symmetric, i.e., $|g| = |g^{-1}|\ \forall\, g \in G$;
- *subadditive* if $|g_1 g_2| \le |g_1 + |g_2|\ \forall\, g_1, g_2 \in G$;
- *finite* if all gauge sets are finite;
- *temperate* if it is finite and the gauge sets grow at most exponentially: $\sup_k \frac{1}{k}\log\operatorname{card}\mathcal{G}_k < \infty$.

A family of gauges $\mathcal{G}^\alpha$ is *uniformly temperate* if $\sup_{\alpha,k}\frac{1}{k}\log\operatorname{card}\mathcal{G}^\alpha_k < \infty$. Clearly, the family of *translations* $g\mathcal{G} = (g\mathcal{G}_k)$, $g \in G$ of any temperate gauge is uniformly temperate.

The gauges considered below are not assumed to be finite nor subadditive unless otherwise specified.

An important class of gauges consists of *word gauges* [Gu80a], i.e., such gauges $(\mathcal{G}_k)$ that $\mathcal{G}_1$ is a set generating G as a semigroup, and $\mathcal{G}_k = (\mathcal{G}_1)^k$ is the set of words of *length* $\le k$ in the alphabet $\mathcal{G}_1$. Any word gauge is subadditive. It is symmetric iff the set $\mathcal{G}_1$ is symmetric, and finite iff $\mathcal{G}_1$ is finite. In the latter case the gauge is temperate. Any two finite word gauges $\mathcal{G}, \mathcal{G}'$ on a finitely generated group G are *equivalent* (quasi-isometric) in the sense that there is a constant $C > 0$ such that

$$\frac{1}{C}|g|_{\mathcal{G}'} \le |g|_{\mathcal{G}} \le C|g|_{\mathcal{G}'} \qquad \forall\, g \in G\ .$$

Thus, for a probability measure μ on a finitely generated group G finiteness of its *first moment* $\sum_g |g|\mu(g)$, or of its *first logarithmic moment* $\sum_g \log|g|\mu(g)$ are invariant properties of the measure μ, being independent of the choice of a finite word gauge $|\cdot|$ on G.

2.2.2. Lemma. *If $\mathcal{G}$ is a temperate gauge on a countable group G, then any probability measure μ with finite first moment with respect to $\mathcal{G}$ also has finite entropy $H(\mu)$.*

Proof (cf. [De86]). Let $\pi = (\pi_k)$ be the projection of the measure μ onto $\mathbb{Z}_+$ determined by the map $g \mapsto |g|$, and α_k be the normalized restrictions of the measure μ onto the sets $D_k = \mathcal{G}_k \setminus \mathcal{G}_{k-1}$, so that $\mu = \sum \pi_k \alpha_k$. Then

$$H(\mu) = H(\pi) + \sum_k \pi_k H(\alpha_k)\ .$$

By standard properties of the entropy

$$\begin{aligned}\sum_k \pi_k H(\alpha_k) &\le \sum_k \pi_k \log\operatorname{card} D_k \le \sum_k \pi_k \log\operatorname{card}\mathcal{G}_k \\ &\le Const \sum_k k\pi_k = Const \sum_g |g|\mu(g) < \infty\ .\end{aligned}$$

On the other hand, monotonicity of the function $t \mapsto -t\log t$ on the interval $[0, e^{-1}]$ implies that

$$H(\pi) = \sum_k (-\log\pi_k)\pi_k \le \sum_k \max\{k, -\log\pi_k\}\pi_k \le \sum_k k\pi_k + \sum_k ke^{-k} < \infty\ .$$

□

2.2.3. For subadditive gauges the Kingman Subadditive Ergodic Theorem immediately implies (cf. [Gu80a], [De80]):

Lemma. *If $\mathcal{G}$ is a subadditive gauge on a countable group G, then for any probability measure μ on G with finite first moment with respect to $\mathcal{G}$ the limit (rate of escape)*

$$\ell(G, \mu, \mathcal{G}) = \lim_{n\to\infty} \frac{|x_n|_{\mathcal{G}}}{n}$$

exists for $\mathbf{P}$-a.e. *sample path $\{x_n\}$ and in the space $L^1(\mathbf{P})$.*

2.3. Ray approximation.

2.3.1. Theorem. *Let μ be a probability measure with finite entropy $H(\mu)$ on a countable group G, and $(B,\lambda)\cong(\Gamma_\xi,\nu_\xi)$ – a μ-boundary. Denote by $\Phi=\mathbf{bnd}_\xi$ the projection from the path space $(G^{\mathbb{Z}_+},\mathbf{P})$ to (B,λ). If for λ-a.e. point $b\in B$ there exists a sequence of uniformly temperate gauges $\mathcal{G}^n=\mathcal{G}^n(b)$ such that*

$$\frac{1}{n}|x_n|_{\mathcal{G}^n(\Phi\boldsymbol{x})}\to 0 \tag{2.3.1}$$

for $\mathbf{P}$-a.e. sample path $\boldsymbol{x}=\{x_n\}$, then (B,λ) is the Poisson boundary of the pair (G,μ).

Proof. Condition (2.3.1) is equivalent to saying that $|x_n|_{\mathcal{G}^n(b)}/n\to 0$ for λ-a.e. $b\in B$ and $\mathbf{P}^b$-a.e. sample path of the random walk conditioned by b (see Theorem 1.4.6). Thus, for λ-a.e. $b\in B$ and any $\varepsilon>0$ there exists a sequence of sets $A_n=A_n(b,\varepsilon)\subset G$ such that

$$\log\operatorname{card}A_n=o(n)\;, \qquad \mathbf{P}^b[x_n\in A_n]\ge\varepsilon\;.$$

Therefore (B,λ) is the Poisson boundary by Theorem 1.7.6. □

2.3.2. Let now $\pi_n: B\to G$ be a sequence of measurable maps from a μ-boundary B to the group G. Geometrically, one can think about the sequences $\pi_n(b)$, $b\in B$ as *"rays"* in G corresponding to points from B. Taking in Theorem 2.3.1 $\mathcal{G}^n(b)=\pi_n(b)\mathcal{G}$, where $\mathcal{G}$ is a fixed temperate gauge on G, we obtain

Theorem. *Let μ be a probability measure with finite entropy $H(\mu)$ on a countable group G, and $(B,\lambda)=\Phi(G^{\mathbb{Z}_+},\mathbf{P})$ – a μ-boundary. If there exist a temperate gauge $\mathcal{G}$ and a sequence of measurable maps $\pi_n: B\to G$ such that*

$$\frac{1}{n}\left|\big(\pi_n(\Phi\boldsymbol{x})\big)^{-1}x_n\right|_{\mathcal{G}}\to 0$$

for $\mathbf{P}$-a.e. sample path $\boldsymbol{x}=\{x_n\}$, then (B,λ) is the Poisson boundary of the pair (G,μ).

Taking $\pi_n(b)\equiv e$ for the one-point μ-boundary we get the following well known result (e.g., see [KV83]).

Corollary. *If μ is a probability measure with finite entropy $H(\mu)$ on a countable group G, and $\ell(G,\mu,\mathcal{G})=0$ for a certain temperate gauge $\mathcal{G}$, then the Poisson boundary of the pair (G,μ) is trivial.*

Remarks. 1. Using Theorem 1.7.6 one can easily show that a.e. convergence in condition (2.3.1) from Theorem 2.3.1 (and in the corresponding condition from Theorem 2.3.2) can be replaced with a weaker convergence in probability:

$$\limsup_{n\to\infty}\mathbf{P}^b\big[|x_n|/n\le\varepsilon\big]>0 \qquad \forall\,\varepsilon>0\;.$$

2. If G is finitely generated, and μ is symmetric and finitely supported, then the condition $\ell(G,\mu,\mathcal{G})=0$ for a certain ($\equiv$ any) word gauge on G is in fact equivalent to triviality of the Poisson boundary $\Gamma(G,\mu)$ as it was proved in [Va85] by using Gaussian estimates for transition probabilities.

2.4. Strip approximation.

2.4.1. We have defined the path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ (see **1.1.4**) as the image of the space of independent μ-distributed increments $\{h_n\}$, $n \ge 1$ under the map

$$x_n = \begin{cases} e, & n = 0 \\ x_{n-1} h_n, & n \ge 1 \,. \end{cases} \tag{2.4.1}$$

Extending the relation $x_n = x_{n-1} h_n$ to all indices $n \in \mathbb{Z}$ (and always assuming that $x_0 = e$) we obtain the measure space $(G^{\mathbb{Z}}, \overline{\mathbf{P}})$ of *bilateral paths* $\overline{\boldsymbol{x}} = \{x_n, \ n \in \mathbb{Z}\}$ corresponding to bilateral sequences of independent μ-distributed increments $\{h_n\}$, $n \in \mathbb{Z}$. For negative indices n formula (2.4.1) can be rewritten as

$$x_{-n} = x_{-n+1} h_{-n+1}^{-1} \,, \qquad n \ge 0 \,,$$

so that

$$\check{x}_n = x_{-n} = h_0^{-1} h_{-1}^{-1} \cdots h_{-n+1}^{-1} \,, \qquad n \ge 0$$

is a sample path of the random walk on G governed by the *reflected measure* $\check{\mu}(g) = \mu(g^{-1})$. The unilateral paths $\boldsymbol{x} = \{x_n\}$, $n \ge 0$ and $\check{\boldsymbol{x}} = \{\check{x}_n\} = \{x_{-n}\}$, $n \ge 0$ are independent, i.e., the map $\overline{\boldsymbol{x}} \mapsto (\boldsymbol{x}, \check{\boldsymbol{x}})$ is an isomorphism of the measure spaces $(G^{\mathbb{Z}}, \overline{\mathbf{P}})$ and $(G^{\mathbb{Z}_+}, \mathbf{P}) \times (G^{\mathbb{Z}_+}, \check{\mathbf{P}})$, where $\check{\mathbf{P}}$ is the measure in the space of unilateral sample paths of the random walk $(G, \check{\mu})$.

2.4.2. Denote by $\overline{U}$ the measure preserving transformation of the space of bilateral paths $(G^{\mathbb{Z}}, \overline{\mathbf{P}})$ induced by the bilateral Bernoulli shift in the space of increments. It is the natural extension of the transformation U of the unilateral path space $(G^{\mathbb{Z}_+}, \mathbf{P})$ defined in **1.2.6** and acts by the same formula (1.2.6) extended to all indices $n \in \mathbb{Z}$: for any $k \in \mathbb{Z}$

$$(\overline{U}^k \overline{\boldsymbol{x}})_n = x_k^{-1} x_{n+k} \qquad \forall n \in \mathbb{Z} \,, \tag{2.4.2}$$

i.e., the path $\overline{U}^k \overline{\boldsymbol{x}}$ is obtained from the path $\overline{\boldsymbol{x}}$ by translating it both in time (by k) and in space (by multiplying by x_k^{-1} on the left in order to satisfy the condition $(\overline{U}^k \overline{\boldsymbol{x}})_0 = e$). In terms of the unilateral paths $\boldsymbol{x}$ and $\check{\boldsymbol{x}}$ applying $\overline{U}^k$ consists (for $k > 0$) in canceling first k factors $x_k = h_1 h_2 \cdots h_k$ from the products $x_n = h_1 h_2 \cdots h_k \cdots h_n$, $n > 0$ (i.e., in applying to $\boldsymbol{x}$ the transformation U^k) and adding on the left k factors $x_k^{-1} = h_k^{-1} \cdots h_2^{-1} h_1^{-1}$ to the products $\check{x}_n = x_{-n} = h_0^{-1} h_{-1}^{-1} \cdots h_{-n+1}^{-1}$:

$$\underbrace{\overbrace{\cdots, h_{-1}, h_0,} \overbrace{h_1, \cdots, h_{k-1}, h_k,}} \underbrace{h_{k+1}, \cdots} \,.$$

2.4.3. Denote by $\check{\Gamma}$ the Poisson boundary of the measure $\check{\mu}$, and by $\check{\nu}$ the corresponding harmonic measure, i.e., the image of the measure $\check{\mathbf{P}}$ under the quotient map from $G^{\mathbb{Z}_+}$ to $\check{\Gamma}$.

Theorem. *The action of the group G on the product $\check\Gamma \times \Gamma$ of the Poisson boundaries of the measures $\check\mu$ and μ is ergodic with respect to the product of harmonic measures $\check\nu \otimes \nu$.*

Proof. Denote by π the measure preserving projection $\overline{\boldsymbol{x}} \mapsto (\check{\boldsymbol{x}}, \boldsymbol{x}) \mapsto (\mathbf{bnd}\,\check{\boldsymbol{x}}, \mathbf{bnd}\,\boldsymbol{x})$ from the bilateral path space $(G^{\mathbb{Z}}, \overline{\mathbf{P}})$ to the product space $(\check\Gamma \times \Gamma, \check\nu \otimes \nu)$. Then as it follows from formula (2.4.2), for any $k \in \mathbb{Z}$

$$\pi(\overline{U}^k \overline{\boldsymbol{x}}) = x_k^{-1} \pi(\overline{\boldsymbol{x}}) \tag{2.4.3}$$

(cf. Lemma 1.2.6). Now, if $A \subset \check\Gamma \times \Gamma$ is a G-invariant subset of $\check\Gamma \times \Gamma$ with $0 < \check\nu \otimes \nu(A) < 1$, then by (2.4.3) the preimage $\pi^{-1}(A)$ is $\overline{U}$-invariant with $0 < \overline{\mathbf{P}}(\pi^{-1}A) = \check\nu \otimes \nu(A) < 1$, which is impossible by ergodicity of the bilateral Bernoulli shift $\overline{U}$. □

2.4.4. If $\operatorname{sgr}(\mu) = G$, then the measure ν belongs to the harmonic measure type, so that Theorem 2.4.3 implies ergodicity of the action of G with respect to the product $[\check\nu_m] \otimes [\nu_m]$ of harmonic measure types on the product of the Poisson boundaries of the measures $\check\mu$ and μ. However, this is no longer true under the weaker condition $\operatorname{gr}(\mu) = G$. The simplest counterexample is described in **1.2.9**. Indeed, in this situation concatenation (with possible cancellation of finite boundary segments) of (right) infinite words from the Poisson boundary of μ and (left) inverse infinite words from the Poisson boundary of $\check\mu$ gives bilateral infinite words such that all but a finite number of their letters are a or b. Then one can easily see that the ergodic components of the G-action with respect to $[\check\nu_m] \otimes [\nu_m]$ are parameterized by maximal finite subwords whose initial and final letters are a^{-1} or b^{-1}.

2.4.5. Theorem. *Let μ be a probability measure with finite entropy $H(\mu)$ on a countable group G, and let (B_-, λ_-) and (B_+, λ_+) be $\check\mu$- and μ-boundaries, respectively. If there exist a gauge $\mathcal{G} = (\mathcal{G}_k)$ on the group G with gauge function $|\cdot| = |\cdot|_{\mathcal{G}}$ and a measurable G-equivariant map S assigning to pairs of points $(b_-, b_+) \in B_- \times B_+$ non-empty "strips" $S(b_-, b_+) \subset G$ such that for all $g \in G$ and $\lambda_- \otimes \lambda_+$-a.e. $(b_-, b_+) \in B_- \times B_+$*

$$\frac{1}{n} \log \operatorname{card}\left[S(b_-, b_+)g \cap \mathcal{G}_{|x_n|}\right] \underset{n\to\infty}{\longrightarrow} 0 \tag{2.4.4}$$

in probability with respect to the measure $\mathbf{P}$ in the space of sample paths $\boldsymbol{x} = \{x_n\}_{n \ge 0}$, then the boundary (B_+, λ_+) is maximal.

Proof. Denote by $\Phi_- : \overline{\boldsymbol{x}} \mapsto \check{\boldsymbol{x}} \mapsto \mathbf{bnd}_{\xi}\, \check{\boldsymbol{x}}$ and $\Phi_+ : \overline{\boldsymbol{x}} \mapsto \boldsymbol{x} \mapsto \mathbf{bnd}_{\xi}\, \boldsymbol{x}$ the projections of the bilateral path space $(G^{\mathbb{Z}}, \overline{\mathbf{P}})$ onto the boundaries $(B_-, \lambda_-) \cong (\check\Gamma_{\xi}, \check\nu_{\xi})$ and $(B_+, \lambda_+) \cong (\Gamma_\xi, \nu_\xi)$, respectively (cf. the proof of Theorem 2.4.3). Replacing if necessary the map S with its appropriate right translation $(b_-, b_+) \mapsto S(b_-, b_+)g$, we may assume without loss of generality that

$$\lambda_- \otimes \lambda_+ \big\{(b_-, b_+) : e \in S(b_-, b_+)\big\} = \overline{\mathbf{P}}\big[e \in S(\Phi_- \overline{\boldsymbol{x}}, \Phi_+ \overline{\boldsymbol{x}})\big] = p > 0\,.$$

Since the map $(b_-, b_+) \mapsto S(b_-, b_+) \subset G$ is G-equivariant, and using formula (2.4.3) in combination with the fact that the measure $\overline{\mathbf{P}}$ is $\overline{U}$-invariant, we then have for any $n \in \mathbb{Z}$

$$
\begin{aligned}
\overline{\mathbf{P}}\big[x_n \in S(\Phi_-\overline{\boldsymbol{x}}, \Phi_+\overline{\boldsymbol{x}})\big] &= \overline{\mathbf{P}}\big[e \in x_n^{-1} S(\Phi_-\overline{\boldsymbol{x}}, \Phi_+\overline{\boldsymbol{x}})\big] \\
&= \overline{\mathbf{P}}\big[e \in S(x_n^{-1}\Phi_-\overline{\boldsymbol{x}}, x_n^{-1}\Phi_+\overline{\boldsymbol{x}})\big] \\
&= \overline{\mathbf{P}}\big[e \in S(\Phi_-\overline{U}^n\overline{\boldsymbol{x}}, \Phi_+\overline{U}^n\overline{\boldsymbol{x}})\big] \\
&= \overline{\mathbf{P}}\big[e \in S(\Phi_-\overline{\boldsymbol{x}}, \Phi_+\overline{\boldsymbol{x}})\big] = p \,.
\end{aligned} \tag{2.4.5}
$$

Since the image of the measure $\overline{\mathbf{P}}$ under the map $\overline{\boldsymbol{x}} \mapsto (\Phi_-\overline{\boldsymbol{x}}, \Phi_+\overline{\boldsymbol{x}})$ is $\lambda_- \otimes \lambda_+$, formula (2.4.5) can be rewritten as

$$
\int\!\!\int p_n^{b_+}\big[S(b_-, b_+)\big]\, d\lambda_-(b_-) d\lambda_+(b_+) = p \,, \tag{2.4.6}
$$

where $p_n^{b_+}$ is the one-dimensional distribution of the conditional measure $\mathbf{P}^{b_+}$ at time n.

Let

$$
K_n = \min\big\{k \ge 1 : \mu_n(\mathcal{G}_k) \ge 1 - p/2\big\} \,,
$$

so that

$$
\mathbf{P}\big[|x_n| \le K_n\big] = \mu_n(\mathcal{G}_{K_n}) \ge 1 - p/2 \,,
$$

or, after conditioning by $\Phi_+\overline{\boldsymbol{x}}$,

$$
\int p_n^{b_+}\big[\mathcal{G}_{K_n}\big]\, d\lambda_+(b_+) \ge 1 - p/2 \,. \tag{2.4.7}
$$

Since for all $(b_-, b_+) \in B_- \times B_+$

$$
p_n^{b_+}\big[S(b_-, b_+) \cap \mathcal{G}_{K_n}\big] \ge p_n^{b_+}\big[S(b_-, b_+)\big] + p_n^{b_+}\big[\mathcal{G}_{K_n}\big] - 1 \,,
$$

(2.4.6) and (2.4.7) imply

$$
\int\!\!\int p_n^{b_+}\big[S(b_-, b_+) \cap \mathcal{G}_{K_n}\big]\, d\lambda_-(b_-) d\lambda_+(b_+) \ge p/2 \,,
$$

whence

$$
\lambda_- \otimes \lambda_+ \Big\{(b_-, b_+) : p_n^{b_+}\big[S(b_-, b_+) \cap \mathcal{G}_{K_n}\big] \ge p/4\Big\} \ge p/4 \,. \tag{2.4.8}
$$

On the other hand, condition (2.4.4) implies that

$$
\frac{1}{n} \log \operatorname{card}\big[S(b_-, b_+) \cap \mathcal{G}_{K_n}\big] \underset{n\to\infty}{\longrightarrow} 0 \qquad \lambda_- \otimes \lambda_+\text{-a.e. } (b_-, b_+) \in B_- \times B_+ \,,
$$

whence there exist a subset $Z \subset B_- \times B_+$ and a sequence φ_n with $\log \varphi_n / n \to 0$ such that

$$
\lambda_- \otimes \lambda_+(Z) \ge 1 - p/8 \,, \tag{2.4.9}
$$

and

$$\text{card}\,\big[S(b_-,b_+)\cap\mathcal{G}_{K(n)}\big]\le\varphi_n \qquad \forall\,(b_-,b_+)\in Z\ . \tag{2.3.8}$$

Combining (2.4.8), (2.4.9) and (2.3.8) shows that there exists a sequence of sets $X_n\subset B_-\times B_+$ such that

$$\begin{aligned}\lambda_-\otimes\lambda_+(X_n)&\ge p/8\ ,\\ p_n^{b_+}\big[S(b_-,b_+)\cap\mathcal{G}_{K_n}\big]&\ge p/4 \qquad \forall\,(b_-,b_+)\in X_n\ ,\\ \text{card}\,\big[S(b_-,b_+)\cap\mathcal{G}_{K_n}\big]&\le\varphi_n \qquad \forall\,(b_-,b_+)\in X_n\ .\end{aligned}$$

Thus, taking Y_n to be the projection of X_n to B_+, we have that $\lambda_+(Y_n)\ge p/8$, and for a.e. $b_+\in Y_n$ there exists a set $A=A(b_+,n)$ with $p_n^{b_+}(A)\ge p/4$ and $\text{card}\,A\le\varphi_n$, so that the boundary (B_+,λ_+) is maximal by Theorem 1.7.6. □

Remarks. 1. Convergence for a.e. (b_-,b_+) in condition (2.4.4) of the Theorem can be replaced with convergence in probability (see above the Remark after Theorem 2.3.2).

2. Formally, this Theorem does not say anything about maximality of the boundary (B_-,λ_-) of the reflected measure $\check\mu$. However, in concrete situations condition (2.4.4), as a rule, can be verified for the measures μ and $\check\mu$ simultaneously (for example, if the gauge $\mathcal{G}$ is symmetric), so that in this case we get maximality of the *both* boundaries (B_-,λ_-) and (B_+,λ_+) simultaneously (see Theorem 2.4.6 below).

2.4.6. Subexponentiality of the intersections $\big[S(b_-,b_+)\cap\mathcal{G}_{|x_n|}\big]$ is the key condition of Theorem 2.4.5. Thus, the "thinner" are the strips $S(b_-,b_+)$ themselves, the larger is the class of measures satisfying condition (2.4.4) of Theorem 2.4.5 (i.e., sample paths $\{x_n\}$ may be allowed to go to infinity "faster"). We shall illustrate this trade-off by giving two more operational corollaries of Theorem 2.4.5.

Theorem. *Suppose that $\mathcal{G}$ is a subadditive temperate gauge on a countable group G with gauge function $|\cdot|=|\cdot|_{\mathcal{G}}$ (particular case: G is finitely generated, and $\mathcal{G}$ is a finite word gauge), and μ is a probability measure on G. Let (B_-,λ_-) and (B_+,λ_+) be $\check\mu$- and μ-boundaries, respectively, and there exists a measurable G-equivariant map $B_-\times B_+\ni(b_-,b_+)\mapsto S(b_-,b_+)\subset G$. If either*

(a) *The measure μ has a finite first moment $\sum|g|\mu(g)$, and for $\lambda_-\otimes\lambda_+$-a.e. (b_-,b_+)*

$$\frac{1}{k}\log\text{card}\,\big[S(b_-,b_+)\cap\mathcal{G}_k\big]\to 0$$

(the strips $S(b_-,b_+)$ grow subexponentially *with respect to $\mathcal{G}$);*

or

(b) *The measure μ has a finite first logarithmic moment $\sum\log|g|\mu(g)$ and finite entropy $H(\mu)$, and for $\lambda_-\otimes\lambda_+$-a.e. (b_-,b_+)*

$$\sup_k\frac{1}{\log k}\log\text{card}\,\big[S(b_-,b_+)\cap\mathcal{G}_k\big]<\infty$$

(the strips $S(b_-,b_+)$ grow polynomially*);*

then the boundaries (B_-, λ_-) *and* (B_+, λ_+) *are maximal.*

Proof. (a) By Lemma 2.2.2, the measure μ has finite entropy, and by Lemma 2.2.3 there exists the rate of escape $\ell(G, \mu, \mathcal{G})$. Now, for any $g \in G$

$$\begin{aligned} \operatorname{card}\left[S(b_-, b_+)g \cap \mathcal{G}_{|x_n|}\right] &= \operatorname{card}\left[S(b_-, b_+) \cap \mathcal{G}_{|x_n|}g^{-1}\right] \\ &\le \operatorname{card}\left[S(b_-, b_+) \cap \mathcal{G}_{|x_n|+|g^{-1}|}\right], \end{aligned}$$

whence condition (2.4.4) is satisfied.

(b) The proof is analogous to the proof of part (a), except for now we have to show that $\log|x_n|/n \to 0$. Indeed,

$$|x_n| = |h_1 h_2 \cdots h_n| \le |h_1| + |h_2| + \cdots + |h_n|,$$

where h_n are the independent μ-distributed increments of the random walk. Since the measure μ has a finite first logarithmic moment, a.e. $\log|h_n|/n \to 0$, which implies that a.e. $\log|x_n|/n \to 0$. Now, for $\lambda_- \otimes \lambda_+$-a.e. (b_-, b_+) and $\mathbf{P}$-a.e. path $\{x_n\}$

$$\begin{aligned} \frac{1}{n}\log\operatorname{card}\left[S(b_-, b_+)g \cap \mathcal{G}_{|x_n|}\right] &\le \frac{1}{n}\log\operatorname{card}\left[S(b_-, b_+) \cap \mathcal{G}_{|x_n|+|g^{-1}|}\right] \\ &= \frac{\log(|x_n| + |g^{-1}|)}{n} \cdot \frac{\log\operatorname{card}\left[S(b_-, b_+) \cap \mathcal{G}_{|x_n|+|g^{-1}|}\right]}{\log(|x_n| + |g^{-1}|)} \to 0. \end{aligned}$$

□

2.4.7. Let us introduce the following condition on a group compactification $\overline{G} = G \cup \partial G$.

(CG) There exists a left-invariant metric d on G such that the corresponding gauge $|\cdot|_d$ on G is temperate and for any two distinct points $b_- \ne b_+ \in \partial G$
 (i) The pencil $P(b_-, b_+)$ of all d-geodesics α in G such that b_- (resp., b_+) is a limit point of the negative (resp., positive) ray of α is non-empty;
 (ii) There exists a finite set $A = A(b_-, b_+)$ such that any geodesic from the pencil $P(b_-, b_+)$ intersects $A(b_-, b_+)$.

Combining Theorems 2.1.4 and 2.4.6 then gives

Theorem. *Let* $\overline{G} = G \cap \partial G$ *be a separable compactification of a countable group* G *satisfying conditions* (CP), (CS), (CG), *and* μ *– a probability measure on* G *such that*

(i) *The subgroup* $\operatorname{gr}(\mu)$ *generated by its support is non-elementary with respect to this compactification;*

(ii) *The measure* μ *has a finite entropy* $H(\mu)$*;*

(iii) *The measure* μ *has a finite first logarithmic moment with respect to the gauge determined by the metric* d *from condition* (CG).

Then the compactification $\overline{G}$ *is* μ*-maximal in the sense that* $\mathbf{P}$*-a.e. sample path* $\boldsymbol{x} = \{x_n\}$ *converges to a limit* $x_\infty = \mathbf{bnd}\,\boldsymbol{x} \in \partial G$*. The limit measure* $\lambda = \mathbf{bnd}(\mathbf{P})$ *is* *the unique μ-stationary probability measure on ∂G, and the measure space $(\partial G, \lambda)$ is the Poisson boundary of the measure μ.*

Proof. Theorem 2.1.4 yields uniqueness of the measure $\lambda = \lambda_+$ and convergence, which implies that $(\partial G, \lambda_+)$ is a μ-boundary. We shall deduce maximality of this boundary from Theorem 2.4.6. Indeed, the reflected measure $\check{\mu}$ satisfies conditions of Theorem 2.1.4 simultaneously with the measure μ. Let λ_- be the unique $\check{\mu}$-stationary measure on ∂G. Since the measures λ_- and λ_+ are purely non-atomic, the diagonal in $\partial G \times \partial G$ has zero measure $\lambda_- \otimes \lambda_+$, so that by condition (CG) for $\lambda_- \otimes \lambda_+$-a.e. $(b_-, b_+) \in \partial G \times \partial G$ there exists a minimal $M = M(b_-, b_+)$ such that all geodesics from the pencil $P(b_-, b_+)$ intersect a M-ball in G. Obviously, the map $(b_-, b_+) \mapsto M(b_-, b_+)$ is G-invariant, so that it must a.e. take a constant value M_0 by Theorem 2.4.3. Now define the strip $S(b_-, b_+) \subset G$ as the union of *all* balls B of diameter M_0 such that any geodesic from the pencil $P(b_-, b_+)$ passes through B. This map is clearly G-equivariant, and for any geodesic α from $S(b_-, b_+)$ the strip $S(b_-, b_+)$ is contained in the M_0-neighbourhood of α. Thus, the strips $S(b_-, b_+)$ have linear growth, so that conditions of Theorem 2.4.6 are satisfied. □

2.5. Asymptotically dissipative actions.

2.5.1. Definition. Let G be a finitely generated group, and (Ω, m) be a measure space endowed with a measure type preserving action of the group G. Fix a finite generating set, and denote by $|\cdot|$ the corresponding word gauge on G. Given a function $\varphi : \mathbb{Z}_+ \to \mathbb{R}_+$, we shall say that a set $E \subset \Omega$ is *φ-wandering* if for a.e. $\omega \in E$

$$\operatorname{card}\{g \in G : g\omega \in E, |g| \le n\} \le \varphi(n) .$$

The action is called *φ-dissipative* if there is a φ-wandering set E such that $\Omega = \bigcup gE$.

For the function $\varphi \equiv 1$ these definitions coincide with the usual definitions of wandering sets and dissipative actions (e.g., see [Kr85]).

2.5.2. Definition. We shall say that a measure type preserving action of a finitely generated group G is *polynomially* (resp., *subexponentially*) *dissipative* if it is φ-dissipative for the function $\varphi(n) = Cn^\alpha$ for some $C, \alpha > 0$ [resp., for all functions $\varphi(n) = Ce^{\varepsilon n}$ with $\varepsilon > 0$ and sufficiently large $C = C(\varepsilon)$]. This definition clearly does not depend on the choice of the finite word gauge $|\cdot|$ on G.

2.5.3. Any G-equivariant measurable map $\omega \mapsto S(\omega) \subset G$ determines a subset

$$E = \{\omega \in \Omega : e \in S(\omega)\} ,$$

and, conversely,

$$g \in S(\omega) \iff e \in g^{-1}S(\omega) = S(g^{-1}\omega) \iff g^{-1}\omega \in E \iff \omega \in gE ,$$

so that any measurable subset $E \subset \Omega$ determines a G-equivariant map

$$\omega \mapsto S(\omega) = \{g \in G : \omega \in gE\} \subset G ,$$

and there is a natural one-to-one correspondence between subsets $E \subset \Omega$ and G-equivariant maps $\omega \mapsto S(\omega) \subset G$.

2.5.4. Thus, Theorem 2.4.6 can be reformulated in the following way:

Theorem. *Let μ be a probability measure on a finitely generated group G, and let (B_-, λ_-) and (B_+, λ_+) be a $\check{\mu}$- and a μ-boundary, respectively. If either*

(a) *The measure μ has a finite first moment, and the G-action on the product space $(B_- \times B_+, \lambda_- \otimes \lambda_+)$ is subexponentially dissipative;*

or

(b) *The measure μ has a finite first logarithmic moment and finite entropy, and the G-action on $(B_- \times B_+, \lambda_- \otimes \lambda_+)$ is polynomially dissipative;*

then the boundaries (B_-, λ_-) and (B_+, λ_+) are maximal.

3. Applications to concrete groups

3.1. Hyperbolic groups.

3.1.1. Let (X, d) be a proper geodesic metric space with a chosen reference point $o \in X$. For a point $x \in X$ put $|x|_o = d(o, x)$, and denote by

$$(x|y)_o = \frac{1}{2}\big[|x|_o + |y|_o - d(x, y)\big]$$

the *Gromov product* on X. The space (X, d) is called *Gromov hyperbolic* if there exists $\delta > 0$ such that the *δ-ultrametric inequality*

$$(x|y)_o \geq \min\big\{(x|z)_o, (y|z)_o\big\} - \delta$$

is satisfied for all $o, x, y, z \in X$ [Gr87], [CDP90], [GH90]. The *hyperbolic boundary* ∂X of a hyperbolic space X is defined as the space of equivalence classes of asymptotic geodesic rays in X (i.e., those which lie at a finite distance one from another), and the definition of the Gromov product $(\cdot|\cdot)$ can be extended to the case when one or both arguments belong to ∂X. The hyperbolic boundary ∂X is the boundary of the *hyperbolic compactification* of X: a sequence $(x_n) \subset X$ converges in this compactification iff $(x_n|x_m) \to \infty$. For any two points $x \in X, \xi \in \partial X$ there exists a geodesic ray (not necessarily unique!) issued from x and converging to the point ξ (i.e., *joining* x and ξ), and for any two distinct points $\xi_1 \neq \xi_2 \in \partial X$ there exists a bilateral geodesic (once again, not necessarily unique) joining ξ_1 and ξ_2.

3.1.2. A finitely generated group G is called (word) *hyperbolic* if its Cayley graph corresponding to a finite symmetric generating set $K \subset G$ is hyperbolic, or, in other words, if the Gromov product associated with the word gauge $|\cdot| = |\cdot|_K$ is δ-ultrametric for an appropriate constant δ (this property is independent of the choice of K). We choose the identity e as a reference point for a hyperbolic group G, and omit the subscript e in the notations $|\cdot|_e$ and $(\cdot|\cdot)_e$. The boundary ∂G of a hyperbolic group G is endowed with a natural action of the group G. Standard examples of hyperbolic groups are fundamental groups of compact negatively curved manifolds and free products of finite or cyclic groups.

3.1.3. The following conditions are equivalent for a subgroup $G' \subset G$ of a hyperbolic group (see [Gr87], [CDP90], [GH90]):

(1) G' is elementary with respect to the hyperbolic compactification of G in the sense of Definition 2.1.3, i.e., G' fixes a finite subset of ∂G;

(2) The limit set $\partial G' \subset \partial G$ of G' (i.e., the boundary of the closure of G' in the hyperbolic compactification) if finite;

(3) G' is amenable;

(4) G' is either a finite extension of the group $\mathbb{Z}$ (then $\operatorname{card} \partial G' = 2$) or a finite group (then $\operatorname{card} \partial G' = 0$).

We shall now describe the Poisson boundary of a hyperbolic group G (always assuming that G is non-elementary). For the sake of comparison we shall use here both the ray and the strip approximations (Theorems 2.3.2 and 2.4.6, respectively).

3.1.4 Definition. A sequence of points (x_n) in a Gromov hyperbolic space X is called *regular* if there exists a geodesic ray α and a number $l \geq 0$ (the *rate of escape*) such that $d\big(x_n, \alpha(nl)\big) = o(n)$, i.e., if the sequence (x_n) asymptotically follows the ray α. If $l > 0$, then we call (x_n) a *non-trivial* regular sequence.

This notion is an analogue of the well known notion of *Lyapunov regularity* for sequences of matrices (see [Ka89]). The idea of the proof of the following result belongs to T. Delzant. In the case when X is a Cartan–Hadamard manifold with pinched sectional curvature another proof (using the Alexandrov Triangle Comparison Theorem) was given in [Ka85b].

3.1.5. Theorem. *A sequence (x_n) in a Gromov hyperbolic space X is regular iff*

(i) $d(x_n, x_{n+1}) = o(n)$;

(ii) $|x_n|/n \to l \geq 0$.

Proof. Clearly, we just have to prove that (i) and (ii) imply regularity under the assumption that $l > 0$. Then $(x_{n-1}|x_n) = nl + o(n)$, and applying the quasi-metric $\rho(x, y) = \exp(-(x|y))$ (see [GH90]) yields convergence of x_n to a point $x_\infty \in \partial X$ in the hyperbolic compactification of the space X. Now we fix geodesics α_n (resp., α_∞) joining the origin o with the points x_n (resp., x_∞), and denote the points at distance t from the origin on these geodesics by $[x_n]_t = \alpha_n(t)$ (resp., $[x_\infty]_t = \alpha_\infty(t)$).

Choose a positive number $\varepsilon < l/2$, and let

$$N = N(\varepsilon) = \min\{n > 0 : (x_{n-1}|x_n) \geq (l - \varepsilon)n\} .$$

In particular, $|x_n| \ge (l-\varepsilon)n$ for $n \ge N$, so that the truncations $x_n^\varepsilon = [x_n]_{(l-\varepsilon)n}$ are well defined. The points $x_{n-1}^\varepsilon, x_n^\varepsilon$ belong to the sides of the geodesic triangle with vertices o, x_{n-1}, x_n, so that

$$d(x_{n-1}^\varepsilon, x_n^\varepsilon) \le \big| |x_{n-1}^\varepsilon| - |x_n^\varepsilon| \big| + 4\delta = l - \varepsilon + 4\delta \qquad \forall n \ge N$$

because $|x_{n-1}^\varepsilon|, |x_n^\varepsilon| \le (x_{n-1}|x_n)$, and geodesic triangles in X are 4δ-thin [GH90, pp. 38, 41]. Therefore, for any two indices $n, m \ge N$

$$\begin{aligned} d(x_n^\varepsilon, x_m^\varepsilon) &\le |n-m|(l+4\delta) , \\ d(x_n^\varepsilon, x_m^\varepsilon) &\ge \big| |x_n^\varepsilon| - |x_m^\varepsilon| \big| = |n-m|(l-\varepsilon) \ge |n-m|\, l/2 , \end{aligned}$$

which means that the sequence $(x_n^\varepsilon)_{n \ge N}$ is a *quasigeodesic*, and by [GH90, p. 101] there exists a geodesic ray β starting at the point x_N^ε such that $d(x_n^\varepsilon, \beta) \le H$ for any $n \ge N$ and a constant $H = H(\delta, l)$. Since $(x_n|x_n^\varepsilon) = n(l-\varepsilon) \to \infty$, the sequence (x_n^ε) also converges to the point x_∞, so that the geodesic rays β and α_∞ are asymptotic. Thus, $d(x_n^\varepsilon, \alpha_\infty) \le H + 8\delta$ and

$$d(x_n, \alpha_\infty) \le H + 8\delta + d(x_n, x_n^\varepsilon) = H + 8\delta + \big(|x_n| - n(l-\varepsilon)\big)$$

for all sufficiently large n (see [GH90, p. 117]). Since ε can be made arbitrarily small, the claim is proven. □

3.1.6. Let now μ be a probability measure on a word hyperbolic group with a finite first moment. We shall fix a word gauge $|\cdot|$ on G and denote by ℓ the corresponding rate of escape (Lemma 2.2.3). Without loss of generality we may assume that the group $\mathrm{gr}(\mu)$ is non-elementary, hence non-amenable (see **3.1.3**), as otherwise the Poisson boundary (Γ, ν) is trivial (see **1.2.8**). Then $\ell > 0$ by **1.2.8** and Corollary of Theorem 2.3.2. Further, since the measure μ (i.e, the lengths of the increments $|h_n| = |x_{n-1}^{-1} x_n| = d(x_{n-1}, x_n)$) has finite first moment, $d(x_{n-1}, x_n) = o(n)$. Thus, conditions of Theorem 3.1.5 are satisfied, and we obtain

Theorem. *Let μ be a probability measure measure with a finite first moment on a hyperbolic group G such that the group $\mathrm{gr}(\mu)$ is non-elementary. Then a.e. sample path of the random walk (G, μ) is a non-trivial regular sequence in G.*

3.1.7. For all points $\xi \in \partial G$ choose a geodesic ray α_ξ from e to ξ in such way that the map $\xi \mapsto \alpha_\xi$ is measurable (for example, take for α_ξ the lexicographically minimal geodesic ray among all rays joining e and ξ), and let $\pi_n(\xi) = \alpha_\xi([n\ell])$, where ℓ is the rate of escape of the random walk (G, μ) and $[t]$ is the integer part of a number t. Then by Theorem 3.1.6 for **P**-a.e. sample path $\{x_n\}$

$$d\big(x_n, \pi_n(x_\infty)\big) = o(n) ,$$

so that by Theorem 2.3.2 we obtain

Theorem. *Let μ be a probability measure measure with a finite first moment on a hyperbolic group G such that the group $\mathrm{gr}(\mu)$ is non-elementary. Then a.e. sample path of the random walk (G, μ) converges in the hyperbolic compactification, and the hyperbolic boundary ∂G with the resulting limit measure is isomorphic to the Poisson boundary of (G, μ).*

3.1.8. Using the strip approximation instead of the ray approximation allows us to obtain a stronger result in a simpler way.

Proposition. *The hyperbolic compactification of a non-elementary hyperbolic group satisfies conditions* (CP), (CS), (CG) *from* **2.1.1** *and* **2.4.7**.

Proof. Condition (CP) follows immediately from the definition of the hyperbolic compactification. For any two distinct points $\xi_- \neq \xi_+ \in \partial G$ let $S(\xi_-, \xi_+)$ be the union of points from all geodesics in G joining ξ_- and ξ_+. Then condition (CS) is implied by quasi-convexity of geodesic hulls of subsets in the hyperbolic boundary [Gr87, 7.5.A] and condition (CG) follows from the fact that any two geodesics in a hyperbolic space with the same endpoints are within uniformly bounded distance one from another. □

Applying Theorems 2.1.4 and 2.4.7 we then get

3.1.9. Theorem. *Let μ be a probability measure on a hyperbolic group G such that the subgroup $\mathrm{gr}(\mu)$ generated by its support is non-elementary. Then almost all sample paths $\{x_n\}$ converge to a (random) point $x_\infty \in \partial G$, so that ∂G with the resulting limit measure λ is a μ-boundary. The measure λ is the unique μ-stationary probability measure on ∂G.*

3.1.10. Theorem. *Under conditions of Theorem 3.1.9, if the measure μ has finite entropy $H(\mu)$ and finite first logarithmic moment $\sum \mu(g) \log |g|$ (particular case: μ has a finite first moment), then $(\partial G, \lambda)$ is isomorphic to the Poisson boundary of (G, μ).*

Remarks. 1. In the case when G is a free group a proof of Theorem 3.1.9 was first indicated by Margulis and announced in [KV83]. Later the same proof was recovered by Cartwright and Soardi [CS89]. For hyperbolic groups a proof of Theorem 3.1.9 in the case when $\mathrm{gr}(\mu) = G$ is given in [Wo93]. Unlike ours, all these proofs use contractivity of the G-action on ∂G (see **2.1.5**).

2. If the measure μ is finitely supported and $\mathrm{sgr}(\mu) = G$, then the *Martin boundary* of the random walk coincides with the hyperbolic boundary [An90] (earlier results for the free group were obtained by Dynkin and Malyutov [DM61] and by Derriennic [De75]), so that in this case Theorems 3.1.9, 3.1.10 follow from the general Martin theory (cf. **1.3.5**).

3. Theorems 3.1.9, 3.1.10 are easily seen to hold for any discrete discontinuous group of isometries of a Gromov hyperbolic space X (in this case instead of a word gauge on G one should take the gauge induced by the ambient metric on X).

3.2. Groups with infinitely many ends.

3.2.1. For a compact subset K of a locally compact topological space X denote by $\mathcal{E}_K = \mathcal{E}_K(X)$ the set (with the discrete topology) of connected components of the complement $X \setminus K$. For $K_1 \subset K_2$ there is a natural homomorphism $\mathcal{E}_{K_2} \to \mathcal{E}_{K_1}$. The projective limit $\mathcal{E}(X)$ of the spaces $\mathcal{E}_K$ as the compacts K exhaust the set X is called the *space of ends* of X. The corresponding compactification $\overline{X} = X \cup \mathcal{E}(X)$ obtained as the projective limit of the compactifications $X \cup \mathcal{E}_K(X)$ is called the *end compactification* of X. For an end $\omega \in \mathcal{E}(X)$ and a compact set $K \subset X$ denote by $C(\omega, K)$ the connected component of $\overline{X} \setminus K$ containing ω. The sets $C(\omega, K)$ form a basis of the *end topology* in $\overline{X}$ at the point ω.

3.2.2. The space of ends $\mathcal{E}(G)$ of a finitely generated group G is defined as the space of ends of its Cayley graph with respect to a certain finite generating set A. Neither the space $\mathcal{E}(G)$ nor the end compactification $G \cup \mathcal{E}(G)$ depend on the choice of A [St71]. Clearly, any geodesic ray in G converges to an end. Conversely, by standard compactness considerations for any two distinct ends from $\mathcal{E}(G)$ there exists a geodesic (not necessarily unique!) joining these ends.

The simplest example of a group with infinitely many ends is the free group F_d of rank $d \geq 2$. This group is also hyperbolic. However, in general, a hyperbolic group may have trivial space of ends (e.g., the fundamental group of a compact negatively curved manifold), and a group with infinitely many ends need not be hyperbolic (e.g., the free product of two copies of the group $\mathbb{Z}^2$). Nonetheless, groups with infinitely many ends still have important for us common geometric properties with hyperbolic groups.

3.2.3. Lemma. *The end compactification of a finitely generated group with infinitely many ends satisfies conditions* (CP), (CS), (CG) *from* **2.1.1** *and* **2.4.7**.

Proof. Condition (CP) is trivial. For verifying condition (CS) let $S(\omega_1, \omega_2)$ be the union of all geodesics in G with endpoints $\omega_1 \neq \omega_2 \in \mathcal{E}(G)$. Take $\omega_0 \neq \omega_1, \omega_2 \in \mathcal{E}(G)$, then there is a finite set $K \subset G$ such that $C(\omega_0, K) \neq C(\omega_1, K), C(\omega_2, K)$, so that the intersection with $C(\omega_0, K)$ of any geodesic joining points in $C(\omega_1, K)$ and $C(\omega_2, K)$ must be contained in the (finite) union of all geodesic segments with endpoints from K. Finally, (CG) immediately follows from the definition of the space of ends. □

Now Theorems 2.1.4 and 2.4.7 imply

3.2.4. Theorem. *Let G be a finitely generated group with infinitely many ends, and μ – a probability measure such that the subgroup* $\operatorname{gr}(\mu)$ *generated by its support is non-elementary. Then almost all sample paths $\{x_n\}$ of the random walk (G, μ) converge to a (random) end $x_\infty \in \mathcal{E}(G)$, so that the space of ends $\mathcal{E}(G)$ with the resulting limit measure λ is a μ-boundary. The measure λ is the unique μ-stationary probability measure on $\mathcal{E}(G)$.*

3.2.5. Theorem. *Under conditions of Theorem 3.2.4, if the measure μ in addition has finite entropy and finite first logarithmic moment (in particular, if μ has finite first moment), then the space $(\mathcal{E}(G),\lambda)$ is isomorphic to the Poisson boundary of the pair (G,μ).*

Remark. Our proof of Theorems 3.2.4, 3.2.5 is synthetic and does not evoke at all the structure theory of groups with infinitely many ends due to Stallings [St71]. According to this theory any such group is either an amalgamated product over a finite group or an HNN-extension over a finite group. If $\mathrm{gr}\,(\mu)=G$ Theorem 3.2.4 was proved by Woess [Wo93] using contractivity properties of the action of G on the space of ends (cf. Remark 1 in **3.1.10**). For finitely supported measures on free groups Dynkin and Malyutov [DM61] and Derriennic [De75] identified the space of infinite words representing ends of the group) with the Martin boundary (see Remark 2 in **3.1.10**). A particular case of Theorem 3.2.5 when the measure μ is finitely supported and $\mathrm{sgr}\,(\mu)=G$ was proved by Woess [Wo89] by applying the Martin theory methods.

3.3. Fundamental groups of rank 1 manifolds.

3.3.1. Let M be a compact Riemannian manifolds with non-positive sectional curvature, and $\widetilde{M}$ – its universal covering space. Two geodesic rays in $\widetilde{M}$ are called *asymptotically equivalent* if each one lies within a finite distance from the other one (cf. **3.1.1**). The *visibility compactification* of $\widetilde{M}$ is obtained by attaching to $\widetilde{M}$ the space of asymptotic classes of geodesic rays in $\widetilde{M}$ (the *sphere at infinity*): a sequence $x_n\in\widetilde{M}$ is convergent in this compactification iff for a certain ($\equiv$ any) reference point $o\in\widetilde{M}$ the directing vectors of the geodesics (o,x_n) converge [Ba95]. The embedding $g\mapsto go\in\widetilde{M}$ allows one to consider the visibility compactification of $\widetilde{M}$ as a compactification of the fundamental group $\pi_1(M)$.

If $\widetilde{M}$ is *irreducible* (i.e., is not a product of two Cartan–Hadamard manifolds), then by the Rank Rigidity Theorem [Ba95] $\widetilde{M}$ is either a symmetric space of non-compact type with rank at least 2, or $\widetilde{M}$ has a *regular* geodesic σ, i.e., such that there is no non-trivial parallel Jacobi field along σ perpendicular to $\dot\sigma$. In the latter case M is said to have *rank* 1. Note that the sectional curvature of a rank 1 manifold M is not necessarily bounded away from 0, and its universal covering space $\widetilde{M}$ is not necessarily hyperbolic in the sense of Gromov.

3.3.2. Theorem. *Let μ be a probability measure on the fundamental group $G=\pi_1(M)$ of a compact rank 1 Riemannian manifold M such that $\mathrm{sgr}\,(\mu)=G$, and μ has a finite first logarithmic moment and finite entropy. Then a.e. sample path of the random walk (G,μ) converges in the visibility compactification, and the sphere $\partial\widetilde{M}$ with the resulting limit measure λ is isomorphic to the Poisson boundary of the pair (G,μ).*

Proof. Convergence of sample paths was established by Ballmann for an arbitrary probability measure on G with $\mathrm{sgr}\,(\mu)=G$ [Ba89, Theorem 2.2]. Moreover, he also proved that $g_n\lambda\to\delta_\gamma$ weakly in $\widetilde{M}\cup\partial\widetilde{M}$ for any sequence $(g_n)\subset G$ such that $g_n\to\gamma$, i.e., that the Dirichlet problem for μ-harmonic functions with boundary data at $\partial\widetilde{M}$ is solvable [Ba89, Theorem 1.8].

It remains to prove maximality of the μ-boundary $(\partial\widetilde{M}, \lambda)$. Denote by $\lambda_+ = \lambda$ and λ_- the harmonic measures of random walks (G, μ) and $(G, \check{\mu})$, respectively, determined by the embedding $G \hookrightarrow \widetilde{M} \cup \partial\widetilde{M}$. Since M has rank one, the set $\mathcal{R} \subset \partial\widetilde{M} \times \partial\widetilde{M}$ of pairs of endpoints of *regular* bi-infinite geodesics in $\widetilde{M}$ is open non-empty, and for any pair of points $(\xi_-, \xi_+) \in \mathcal{R}$ there is a unique geodesic $\sigma(\xi_-, \xi_+)$ joining these points [Ba95]. Then solvability of the Dirichlet problem and quasi-invariance of the measures λ_-, λ_+ with respect to the action of G (see **1.2.5**) implies that $\lambda_- \times \lambda_+(\mathcal{R}) > 0$, whence $\lambda_- \times \lambda_+(\mathcal{R}) = 1$ by Theorem 2.4.3.

Since the quotient manifold M is compact, there exists a number $d > 0$ such that for any point $x \in \widetilde{X}$ the d-ball centered at x intersects the orbit Go. Then the strips in G defined as

$$S(\xi_-, \xi_+) = \{g \in G : \mathrm{dist}\big(go, \sigma(\xi_-, \xi_+)\big) \le d\}$$

are non-empty, and the map $(\xi_-, \xi_+) \mapsto S(\xi_-, \xi_+)$ is G-equivariant (here dist is the Riemannian metric on $\widetilde{M}$). The gauge $|g| = \mathrm{dist}(o, go)$ on G is temperate and subadditive, and, since M is compact, it is equivalent to any finite word gauge on G [Mi68], so that the measure μ has finite first logarithmic moment with respect to $|\cdot|$. Clearly, all strips $S(\xi_-, \xi_+)$ (being neighbourhoods of geodesics) have linear growth with respect to the gauge $|\cdot|$, and conditions of Theorem 2.4.6 (b) are satisfied. □

Remarks. 1. For measures μ with a finite first moment Theorem 3.3.2 was first proved by Ballmann and Ledrappier [BL94]. Our proof goes along the same lines, except for the fact that using Theorem 2.4.6 allows us to obtain the result in greater generality and to avoid at the same time tedious dimension estimates (Section 3 in [BL94]).

2. Theorem 3.3.2 together with the Rank Rigidity Theorem and the identification of the Poisson boundary for discrete subgroups of semi-simple Lie groups (Section 3.4) implies a description of the Poisson boundary for the fundamental group of *any* compact non-positively curved Riemannian manifold.

3.4. Discrete subgroups of semi-simple Lie groups.

3.4.1. Let $\mathcal{G}$ be a connected semi-simple real Lie group with finite center, $\mathcal{K}$ – its maximal compact subgroup, and $S = \mathcal{G}/\mathcal{K}$ – the corresponding Riemannian symmetric space with the origin $o \cong \mathcal{K}$. Fix a dominant Weyl chamber $\mathfrak{A}^+$ in the Lie algebra $\mathfrak{A}$ of a Cartan subgroup $\mathcal{A}$, and denote by $\mathfrak{A}_1^+$ (resp., by $\overline{\mathfrak{A}_1^+}$) the intersection of $\mathfrak{A}^+$ (resp., of its closure $\overline{\mathfrak{A}^+}$) with the unit sphere of the Euclidean distance $\|\cdot\|$ determined by the Killing form $\langle\cdot,\cdot\rangle$. Any point $x \in S$ can be presented as $x = k(\exp a)o$, where $k \in \mathcal{K}$, and $a = r(x) \in \overline{\mathfrak{A}^+}$ is the uniquely determined *radial part* of x. Then the Riemannian distance $\mathrm{dist}(o, x)$ from o to x equals $\|r(x)\|$.

3.4.2. Denote by ∂S the boundary (the *sphere at infinity*) of the *visibility compactification* of S (cf. **3.3.1**). We identify points from ∂S with geodesic rays issued from o. The $\mathcal{G}$-orbits in ∂S are parameterized by vectors $a \in \overline{\mathfrak{A}_1^+}$: the orbit ∂S_a consists of the limits of all geodesic rays of the form $\xi(t) = g\exp(ta)o$. Stabilizers of points $\xi \in \partial S$ are parabolic subgroups of $\mathcal{G}$, which are minimal iff $\xi \in \partial S_a, a \in \mathfrak{A}_1^+$. Thus, the orbits ∂S_a corresponding to non-degenerate vectors $a \in \mathfrak{A}_1^+$ are isomorphic to the *Furstenberg boundary* $\mathcal{B} = \mathcal{G}/\mathcal{P}$, where $\mathcal{P} = \mathcal{M}\mathcal{A}\mathcal{N}$ is the minimal parabolic subgroup determined by the Iwasawa decomposition $\mathcal{G} = \mathcal{K}\mathcal{A}\mathcal{N}$ (i.e., $\mathcal{M}$ is the centralizer of $\mathcal{A}$ in $\mathcal{K}$), and the orbits ∂S_a corresponding to vectors a from the walls of the Weyl chamber $\mathfrak{A}^+$ are isomorphic to quotients of the Furstenberg boundary (i.e., to quotients of $\mathcal{G}$ by non-minimal parabolic subgroups) [Ka89]. Moreover, there exists a canonical map $\overline{\mathfrak{A}_1^+} \times \mathcal{B} \to \partial S$ such that $\{a\} \times \mathcal{B} \to \partial S_a$ is one-to-one for $a \in \mathfrak{A}_1^+$ (cf. below **3.4.4**).

3.4.3. We call a sequence of points $x_n \in S$ *regular* if there exists a geodesic ray ξ and a number $l \geq 0$ such that $\operatorname{dist}\big(x_n, \xi(nl)\big) = o(n)$ (cf. Definition 3.1.4). If $l > 0$, then x_n converges in the visibility compactification to the same point as the ray ξ.

Theorem [Ka89]. *A sequence of points x_n in a non-compact Riemannian symmetric space S is regular if and only if* $\operatorname{dist}(x_n, x_{n+1}) = o(n)$ *and there exists a limit* $a = \lim r(x_n)/n \in \overline{\mathfrak{A}^+}$.

Remark. The definition of regular sequences is inspired by the notion of *Lyapunov regularity*, and Theorem 3.4.3 can be used for proving the *Oseledec multiplicative ergodic theorem* and its generalizations [Ka89]. The vector a is called the *Lyapunov vector* of the sequence x_n (cf. below **3.4.10**).

3.4.4. The map $go \mapsto gm$, where m is the unique $\mathcal{K}$-invariant probability measure on $\mathcal{B}$, determines an embedding of S into the space of Borel probability measures on $\mathcal{B}$, which gives rise to the *Furstenberg compactification* of S obtained as the closure of the family of measures $\{gm\}$ in the weak topology. The boundary of this compactification consists (unless S has rank 1) of several G-transitive components, one of which is $\mathcal{B}$ (corresponding to limit δ-measures). If a sequence $x_n \in S$ converges in the visibility compactification to a point b from a non-degenerate orbit $\partial S_a \cong \mathcal{B}$, $a \in \mathfrak{A}_1^+$, then x_n also converges to $b \in \mathcal{B}$ in the Furstenberg compactification [Mo64].

Another definition of the Furstenberg boundary $\mathcal{B}$ (analogous to that of the visibility boundary ∂S) can be given in terms of maximal totally geodesic flat subspaces of S (*flats*) [Mo73]. For a given flat f any basepoint $x \in f$ determines a decomposition of f into *Weyl chambers* of f based at x. Then $\mathcal{B}$ coincides with the space of asymptotic classes of Weyl chambers in S (two chambers are asymptotic if they are within a bounded distance one from the other).

3.4.5. A flat with a distinguished class of asymptotic Weyl chambers is called an *oriented flat*. For an oriented flat $\overline{f}$ denote by $-\overline{f}$ the same flat with the orientation opposite to that of $\overline{f}$, and let $\pi_+(\overline{f}) \in \mathcal{B}$ (resp., $\pi_-(\overline{f}) = \pi_+(-\overline{f})$) be the corresponding asymptotic classes of Weyl chambers (the "endpoints" of $\overline{f}$).

Denote by $\overline{f}_0$ the standard flat $f_0 = \exp(\mathfrak{A})o$ with the orientation determined by $\mathfrak{A}^+$. Let $b_0 = \pi_+(\overline{f}_0)$, and $b_w = \pi_+(w\overline{f}_0), w \in W$, where W is the *Weyl group* which acts simply transitively on orientations of f_0. Denote by w_0 the element of W opposite to the identity which is determined by the relation $w_0\overline{f}_0 = -\overline{f}_0$. Then the Bruhat decomposition of the group $\mathcal{G}$ [Bo69] and transitivity of the action of $\mathcal{G}$ on the space of oriented flats imply

Theorem. *The $\mathcal{G}$-orbits $\mathcal{O}_w = \mathcal{G}(b_0, b_w), w \in W$ determine a stratification of the product $\mathcal{B} \times \mathcal{B}$, and $\mathcal{O}_{w_0}$ is the only orbit of maximal dimension. For any oriented flat $\overline{f}$ the pair of its endpoints $(\pi_-(\overline{f}), \pi_+(\overline{f}))$ belongs to $\mathcal{O}_{w_0}$, and, conversely, for any pair $(b_-, b_+) \in \mathcal{O}_{w_0}$ there exists a unique oriented flat $\overline{f}(b_-, b_+)$ with endpoints (b_-, b_+).*

Remark. In the rank 1 case flats are bilateral geodesics in S, and Weyl chambers are geodesic rays in S. The Weyl group consists of only 2 elements, and the orbits in $\mathcal{B} \times \mathcal{B}$ are the diagonal and its complement.

3.4.6. Theorem. *Let μ be a probability measure on a discrete subgroup $G \subset \mathcal{G}$ of a semi-simple Lie group $\mathcal{G}$ with a finite first moment $\sum \mathrm{dist}(o, go)\mu(g) < \infty$. Then*

(i) **P**-a.e. *sample path $\{x_n\}$ of the random walk (G, μ) is regular, and the Lyapunov vector $a = a(\mu) = \lim r(x_n o)/n \in \overline{\mathfrak{A}}^+$ does not depend on $\{x_n\}$;*
(ii) *If $a \neq 0$, then for* **P**-a.e. *sample path $\{x_n\}$ the sequence $x_n o$ converges in the visibility compactification to a limit point from the orbit ∂S_a;*
(iii) *If $a = 0$, then the Poisson boundary of the pair (G, μ) is trivial, and if $a \neq 0$ it is isomorphic to ∂S_a with the limit measure determined by (ii).*

Proof. Existence of the Lyapunov vector follows from Lemma 2.2.3 applied to matrix norms of finite dimensional representations of $\mathcal{G}$, see [Ka89]. Moreover, finiteness of the first moment of the measure μ implies that **P**-a.e. $\mathrm{dist}(x_n o, x_{n+1} o) = o(n)$ (cf. the proof of Theorem 3.1.6), so that (i) and (ii) follow from Theorem 3.4.3.

Since the growth of S is exponential, the gauge $g \mapsto \mathrm{dist}(o, go)$ on G induced by the Riemannian metric dist is temperate (see Definition 2.2.1), and combining Lemma 2.2.2 and Theorems 2.3.2, 3.4.3 we get (iii). □

Remark. If the group $\mathrm{gr}(\mu)$ generated by the support of μ is non-amenable, then by **1.2.8** the Poisson boundary of (G, μ) is non-trivial, and thereby $a \neq 0$. This observation was first made by Furstenberg [Fu63b].

3.4.7. If the measure μ does not have a finite first moment and the rank of $\mathcal{G}$ is greater than 1, convergence in the visibility compactification does not necessarily hold any more. However, in this situation one can still prove convergence in the Furstenberg compactification by imposing some irreducibility conditions on the group $\mathrm{gr}(\mu)$.

Theorem [GR85]. *Let G be a discrete subgroup of a semi-simple Lie group $\mathcal{G}$, and μ – a probability measure on G such that*

(i) *The semigroup $\mathrm{sgr}(\mu)$ generated by the support of μ contains a sequence g_n such that $\langle \alpha, r(g_n) \rangle \to \infty$ for any positive root α;*
(ii) *No conjugate of the group $\mathrm{gr}(\mu)$ is contained in a finite union of left translations of degenerate double cosets from the Bruhat decomposition of $\mathcal{G}$.*

Then

(j) *For* **P**-a.e. *sample path* $\{x_n\}$ *of the random walk* (G,μ) *the sequence* $x_n o$ *converges in the Furstenberg compactification of the symmetric space* S*;*

(jj) *The corresponding limit measure* λ *is concentrated on the Furstenberg boundary* $\mathcal{B}$*, and it is the unique* μ*-stationary measure on* $\mathcal{B}$*.*

(jjj) *For any point* $b_- \in \mathcal{B}$ *the set* $\{b_+ \in \mathcal{B} : (b_-, b_+) \in \mathcal{O}_{w_0}\}$ *has full measure* λ*, where* $\mathcal{O}_{w_0}$ *is the maximal dimension stratum of the Bruhat stratification in* $\mathcal{B} \times \mathcal{B}$ *defined in* **3.4.5**.

Remark. As it was noticed in [GM89], in the case when $\mathcal{G}$ is an algebraic group conditions (i) and (ii) follow from Zariski density of the semigroup $\operatorname{sgr}(\mu)$ in $\mathcal{G}$. However, these conditions can be also satisfied without $\operatorname{sgr}(\mu)$ being Zariski dense [GR89].

3.4.8. Conditions (i) and (ii) of Theorem 3.4.7 are clearly satisfied simultaneously for the measure μ and for the reflected measure $\check{\mu}$, and by Theorem 3.4.7 (jjj) the product $\lambda_- \times \lambda_+$ of the limit measures of the random walks $(G,\check{\mu})$ and (G,μ) is concentrated on the orbit $\mathcal{O}_{w_0}$. Since the flats in S have polynomial growth, the strips in G defined as

$$S(b_-,b_+) = \{g \in G : \operatorname{dist}(go, \overline{f}(b_-,b_+)) \le R\} ,$$

where $\overline{f}(b_-,b_+)$ is the flat in S with endpoints b_-, b_+, also have polynomial growth (and they are a.e. non-empty for a sufficiently large R). Theorem 2.4.6 (b) then implies

Theorem. *Under conditions of Theorem 3.4.7, if the measure* μ *has finite first logarithmic moment* $\sum \log \operatorname{dist}(go,o)\,\mu(g)$ *and finite entropy* $H(\mu)$*, then the Poisson boundary of* (G,μ) *is non-trivial and it is isomorphic to the Furstenberg boundary* $\mathcal{B}$ *with the limit measure determined by Theorem 3.4.7 (jj).*

3.4.9. *Remarks.* 1. Theorem 3.4.6 was first announced in [Ka85b]. For discrete subgroups of $SL(d,\mathbb{R})$ another proof (under somewhat more restrictive conditions) was independently obtained in [Le85].

2. Conditions of Theorem 3.4.8 on the decay at infinity of the measure μ are more general than those of Theorem 3.4.6. As a trade-off, Theorem 3.4.8 requires irreducibility assumptions (i) and (ii) from Theorem 3.4.7, whereas Theorem 3.4.6 does not impose any conditions at all on the support of the measure μ. Note that if the measure μ has a finite first moment, then under the conditions of Theorem 3.4.7 the vector $a(\mu)$ from Theorem 3.4.6 belongs to $\mathfrak{A}^+$ [GR85], so that the orbit ∂S_a is isomorphic to $\mathcal{B}$, and the descriptions of the Poisson boundary given in Theorems 3.4.6 and 3.4.8 coincide. Actually, Theorem 3.4.7 can be also used for identifying the Poisson boundary for measures with a finite first moment without the irreducibility assumptions (i) and (ii). In this case instead of flats one has to take the symmetric subspaces of S corresponding to pairs of boundary points which are not in general position with respect to the Bruhat decomposition and use the fact that the rate of escape along these subspaces is sublinear (cf. Theorem 3.5.8 below).

3. The limit measure λ on the Furstenberg boundary $\mathcal{B}$ does not have to be absolutely continuous with respect to the Haar measure on $\mathcal{B}$. Namely, for any finitely generated Zariski dense discrete subgroup $G \subset \mathcal{G}$ the author constructed a symmetric finitely supported measure μ on G with $\operatorname{gr}(\mu) = G$ such that λ is singular.

3.4.10. Example. Let $\mathcal{G} = SL(d,\mathbb{R})$ with a maximal compact subgroup $\mathcal{K} = SO(d)$. The map $g\mathcal{K} \mapsto \sqrt{gg^*}$ identifies the symmetric space $S = \mathcal{G}/\mathcal{K}$ with the set of positive definite $d \times d$ matrices with determinant 1, and the origin $o \cong \mathcal{K}$ corresponds to the identity matrix. Under this identification the action of $\mathcal{G}$ on S has the form $(g,x) \mapsto \sqrt{gx^2g^*}$. Take for a Cartan subgroup $\mathcal{A} \subset \mathcal{G}$ the group of diagonal matrices with positive entries, so that its Lie algebra $\mathfrak{A}$ is the space $\{a = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{R}^d : \sum \alpha_i = 0\}$, and choose a dominant Weyl chamber in $\mathfrak{A}$ as $\mathfrak{A}^+ = \{a \in \mathfrak{A} : \alpha_1 > \alpha_2 > \dots \alpha_d\}$. The radial part $r(x) \in \overline{\mathfrak{A}}^+$ of a matrix $x \in S$ is the ordered vector of logarithms of its eigenvalues, and the restriction of the Killing form to $\mathfrak{A}$ is the usual Euclidean form.

Geodesic rays in S starting from o have the form $\xi(t) = \xi_1^t$, where $\xi_1 \in S$ is a matrix at distance 1 from the origin o (i.e., such that $\|r(\xi_1)\| = 1$), so that the visibility boundary ∂S ($\equiv$ the space of geodesic rays issued from o) can be identified with the set S_1 of all such matrices, and a sequence $x_n \in S$ converges in the visibility compactification to $\xi_1 \in S_1 \cong \partial S$ iff $\log x_n / \|r(x_n)\| \to \log \xi_1$.

Matrices $\xi_1 \in S_1$ are parameterized by their eigenvalues and eigenspaces. However, it is more convenient to deal instead with the associated flags in $\mathbb{R}^d$. Namely, let $\lambda_1 > \dots > \lambda_k$ be the distinct coordinates of the vector $r(\xi_1) = a$. Denote the eigenspace and the multiplicity of an eigenvalue λ_i by $E_i \subset \mathbb{R}^d$ and $d_i = \dim E_i$, respectively. Then ξ_1 is uniquely determined by the vector a and the flag $V_1 \subset V_2 \subset \dots \subset V_k = \mathbb{R}^d$, where $V_i = \bigoplus_{j=k-i+1}^k E_j$. The spaces V_i can be described by using the *Lyapunov exponents* $\chi(v) = \lim \log \|\xi_1^t v\|/t$, $v \in \mathbb{R}^d$ of the ray $\xi(t) = \xi_1^t$ as $V_i = \{v : \chi(v) \le \lambda_{k-i+1}\}$ (here and below we assume $\chi(0) = -\infty$).

Thus, for a given vector $a \in \overline{\mathfrak{A}}_1^+$ the corresponding $\mathcal{G}$-orbit $\partial S_a \subset \partial S$ is the variety of flags in $\mathbb{R}^d$ of the type $(d_k, d_{k-1}+d_k, \dots, d_2+d_3+\dots+d_k)$, where d_i are the multiplicities of components of a. The Furstenberg boundary $\mathcal{B} = \mathcal{G}/\mathcal{P}$ of S is isomorphic to non-degenerate orbits ∂S_a, $a \in \mathfrak{A}_1^+$ and coincides with the variety of full flags in $\mathbb{R}^d$, the minimal parabolic subgroup $\mathcal{P}$ being the group of upper triangular matrices.

For $\mathcal{G} = SL(d,\mathbb{R})$ the first moment condition from Theorem 3.4.6 takes the form

$$\sum \log \|g\| \mu(g) < \infty \,, \tag{3.4.1}$$

and part (i) of the Theorem is equivalent to saying that there exists a vector $a \in \overline{\mathfrak{A}}^+$ such that for $\mathbf{P}$-a.e. sample path $\{x_n\}$ the sequence of matrices x_n^* is *Lyapunov regular* with the *Lyapunov spectrum* a (see [Ka89]). Namely, for any $v \in \mathbb{R}^d \setminus \{0\}$ there exists a limit $\chi(v) = \lim \log \|x_n^* v\|/n \in \{\lambda_1 > \dots > \lambda_k\}$, and the subspaces $V_i = \{v \in \mathbb{R}^d : \chi(v) \le \lambda_{k-i+1}\}$ have dimensions $\dim V_i = d_{k-i+1} + \dots + d_k$, where λ_i are the distinct components of a with multiplicities d_i. If $a \neq 0$, then the limit of the sequence $\sqrt{x_n x_n^*}$ in the visibility compactification belongs to the orbit $\partial S_{a/\|a\|}$ and is determined by the *Lyapunov flag* $\{V_i\}$ of the sequence x_n^*. Therefore, Theorem 3.4.6 identifies the Poisson boundary for a measure μ on a discrete subgroup of $SL(d,\mathbb{R})$ satisfying the moment condition (3.4.1) with the space of corresponding Lyapunov flags (the type of these flags is determined by the degeneracy of the Lyapunov spectrum).

The standard flat f_0 in S is the set of diagonal matrices with positive entries, and the positive orientation on it determines the standard flag b_0 consisting of the subspaces

$E_i \oplus \dots \oplus E_d$, where E_i are the coordinate subspaces of $\mathbb{R}^d$. The Weyl group W is isomorphic to the symmetric group of the set $\{1, 2, \dots, d\}$, and it acts on f_0 by permuting the diagonal entries. The element $w_0 \in W$ is the permutation $w_0 : (1, 2, \dots, d-1, d) \mapsto (d, d-1, \dots, 2, 1)$; the flag $b_{w_0} = w_0 b_0$ opposite to b_0 is obtained by reversing the order of coordinates and consists of subspaces $E_1 \oplus \dots \oplus E_i$. For any vector $a \in \mathfrak{A}^+$ the matrices $\exp(ta) \in S$ converge in the Furstenberg compactification to b_0 (resp., to b_{w_0}) when $t \to \infty$ (resp., $t \to -\infty$). More generally, a pair of flags (b_-, b_+) belongs to the $\mathcal{G}$-orbit of maximal dimension $\mathcal{O}_{w_0}$ in $\mathcal{B} \times \mathcal{B}$ iff there exists a matrix $g \in \mathcal{G}$ such that the sequence $g^n o = \left(g^n g^{*n}\right)^{1/2}$ (resp., the sequence $g^{-n} o = \left(g^{-n} g^{*-n}\right)^{1/2}$) converges in the Furstenberg compactification to b_+ (resp., b_-), i.e., iff the spectrum of g is simple, absolute values of its eigenvalues are all pairwise distinct, and the Lyapunov flags of the sequences g^{*n} and g^{*-n} are b_+ and b_-, respectively. In fact, the stratification of $\mathcal{B}$ into the subvarieties $\{b_+ \in \mathcal{B} : (b_-, b_+) \in \mathcal{O}_w\}$ obtained for a fixed $b_- \in \mathcal{B}$ is the well known *Schubert stratification* of the flag variety.

Theorem 3.4.8 allows then to identify the Poisson boundary with the flag variety for any measure μ on a discrete subgroup of $SL(d, \mathbb{R})$ provided that $\operatorname{sgr}(\mu)$ is Zariski dense in $SL(d, \mathbb{R})$, the measure μ satisfies the moment condition $\sum \log\log \|g\| \mu(g) < \infty$ and has a finite entropy $H(\mu)$.

3.5. Polycyclic groups.

3.5.1. A discrete group G is called *polycyclic* if it admits a finite normal series $\{e\} = G_0 \subset G_1 \subset G_2 \subset \dots \subset G_n = G$ with cyclic quotients G_{i+1}/G_i. In a sense, polycyclic groups are "finite dimensional" discrete solvable groups. Indeed, they can be characterized as solvable groups with finitely generated subgroups, or, even more, as solvable groups with finitely generated abelian subgroups; solvable groups of integer matrices are polycyclic, and, conversely, every polycyclic group has a faithful representation in $GL(n, \mathbb{Z})$ [Sg83].

3.5.2. Any semi-direct product $A \leftthreetimes N$ of a finitely generated abelian group A and a finitely generated nilpotent group N is polycyclic. In fact, *all* polycyclic groups can be "essentially" obtained in this way. Before formulating the corresponding result recall that for any finitely generated torsion free nilpotent group N there is a uniquely determined simply connected real nilpotent Lie group $\mathcal{N}$ (the *Lie hull* of N) containing N as a cocompact lattice, and any automorphism of N uniquely extends to an automorphism of $\mathcal{N}$ [Sg83]. An automorphsim of $\mathcal{N}$ is called *semi-simple* if the tangent automorphism of its Lie algebra $\mathfrak{N}$ is diagonalisable in the complexification $\mathfrak{N}^{\mathbb{C}}$.

We shall say that a discrete group G is an $\mathcal{S}$*-group* if it can be presented as a semi-direct product $G = A \leftthreetimes N$ of a finitely generated free abelian group A and a finitely generated torsion free nilpotent group N determined by a semi-simple action of A on N. If a polycyclic group G is contained in an $\mathcal{S}$-group G', then it is called *splittable*, and the embedding $G \hookrightarrow G'$ is called a *semi-simple splitting* of G.

Proposition [Sg83, Theorem 7.2]**.** *Every polycyclic group contains a normal splittable polycyclic subgroup of finite index.*

3.5.3. We shall identify a simply connected real nilpotent Lie group $\mathcal{N}$ with its Lie algebra $\mathfrak{N}$ by using the *Baker–Campbell–Hausdorff multiplication formula* $x \times y = x + y + \frac{1}{2}[x,y] + \dots$. Denote by

$$\mathfrak{N}_1 = \mathfrak{N} \supset \mathfrak{N}_2 = [\mathfrak{N},\mathfrak{N}] \supset \mathfrak{N}_3 = [\mathfrak{N},\mathfrak{N}_2] \supset \dots \supset \mathfrak{N}_{r+1} = \{0\}$$

the *lower central series* of the Lie algebra $\mathfrak{N}$, where r is the *nilpotency class* of $\mathfrak{N}$, and by $\deg x = \max\{l : x \in \mathfrak{N}_l\}$ the corresponding graduation on $\mathfrak{N}$. By $\deg P$ we shall denote the degree of a polynomial P on $\mathfrak{N}$ with respect to the graduation deg. It is well known [Go76] that the group multiplication given by the Baker–Campbell–Hausdorff formula is polynomial, and, moreover, it is *linear in principal terms* with respect to the graduation deg in the following sense: if $\{e_i\}$ is a linear basis in $\mathfrak{N}$ *adapted* to the filtration $\{\mathfrak{N}_l\}$ (i.e., it contains precisely $\dim \mathfrak{N}_l$ vectors from $\mathfrak{N}_l$ for any $l = 1, 2, \dots, r$), then $(x \times y)_i = x_i + y_i + P_i(x,y)$, where P_i is a polynomial with $\deg P_i \le \deg e_i$ and with partial degrees with respect to x and y strictly less than $\deg e_i$.

3.5.4. As it follows from the Baker–Campbell–Hausdorff formula, any automorphism of the Lie algebra $\mathfrak{N}$ is also an automorphism of the Lie group $\mathcal{N} = (\mathfrak{N}, \times)$, and, conversely, any automorphism of the Lie group $\mathcal{N}$ coincides (as a map of $\mathfrak{N}$ onto itself) with the corresponding tangent automorphism of the Lie algebra $\mathfrak{N}$.

Let T be a semi-simple action of a free abelian group $A \cong \mathbb{Z}^d \subset \mathbf{A} \cong \mathbb{R}^d$ on $\mathfrak{N}$, and $\Lambda \subset \mathrm{Hom}\,(A, \mathbb{C}^*)$ be the set of *weights* of the corresponding representation of A in the complexification $\mathfrak{N}^{\mathbb{C}}$. Denote the corresponding *weight subspaces* by

$$\mathfrak{N}^{\mathbb{C}}_\lambda = \{x \in \mathfrak{N}^{\mathbb{C}} : T^a x = \lambda(a)x \quad \forall\, a \in A\} \subset \mathfrak{N}^{\mathbb{C}} ,$$

so that $\mathfrak{N}^{\mathbb{C}} = \bigoplus \mathfrak{N}^{\mathbb{C}}_\lambda$. The functions $\log|\lambda|$, $\lambda \in \Lambda$ are uniquely extendable to homomorphisms from $\mathbf{A}$ to the additive group $\mathbb{R}$, and for a vector $\mathbf{a} \in \mathbf{A}$ denote by $\Lambda_-(\mathbf{a}), \Lambda_0(\mathbf{a}), \Lambda_+(\mathbf{a})$ the sets of *contracting*, *neutral* and *expanding* weights (with respect to $\mathbf{a}$) determined by the sign of $\log|\lambda|(\mathbf{a})$, and let

$$\mathfrak{N}^{\mathbb{C}}_-(\mathbf{a}) = \bigoplus_{\lambda\in\Lambda_-} \mathfrak{N}^{\mathbb{C}}_\lambda , \qquad \mathfrak{N}^{\mathbb{C}}_0(\mathbf{a}) = \bigoplus_{\lambda\in\Lambda_0} \mathfrak{N}^{\mathbb{C}}_\lambda , \qquad \mathfrak{N}^{\mathbb{C}}_+(\mathbf{a}) = \bigoplus_{\lambda\in\Lambda_+} \mathfrak{N}^{\mathbb{C}}_\lambda ,$$

be the corresponding (contracting, neutral and expanding) subspaces of $\mathfrak{N}^{\mathbb{C}}$. Below we shall usually omit the (fixed) vector $\mathbf{a} \in \mathbf{A}$ from our notations.

Proposition. *Let T be a semi-simple action of a free abelian group A on a nilpotent Lie algebra $\mathfrak{N}$. Then*

(i) *The subspaces $\mathfrak{N}^{\mathbb{C}}_-, \mathfrak{N}^{\mathbb{C}}_0, \mathfrak{N}^{\mathbb{C}}_+$ are complexifications of subspaces $\mathfrak{N}_-, \mathfrak{N}_0, \mathfrak{N}_+ \subset \mathfrak{N}$, and $\mathfrak{N} = \mathfrak{N}_- \oplus \mathfrak{N}_0 \oplus \mathfrak{N}_+$.*

(ii) *The subspaces $\mathfrak{N}_-, \mathfrak{N}_0, \mathfrak{N}_+$ and $\mathfrak{N}^*_- = \mathfrak{N}_- \oplus \mathfrak{N}_0, \mathfrak{N}^*_+ = \mathfrak{N}_+ \oplus \mathfrak{N}_0$ can be characterized in the following way: if $(a_k) \subset A$ is a certain ($\equiv$ any) sequence in A such that*

$a_k/k \to \mathbf{a}$, then

$$\begin{aligned}
x \in \mathfrak{N}_- \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{a_k} x\| < 0 \ , \\
x \in \mathfrak{N}_-^* \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{a_k} x\| \le 0 \ , \\
x \in \mathfrak{N}_0 \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{a_k} x\| = 0 \ , \qquad \lim \tfrac{1}{k} \log \|T^{-a_k} x\| = 0 \ , \\
x \in \mathfrak{N}_+^* \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{-a_k} x\| < 0 \ , \\
x \in \mathfrak{N}_-^* \setminus \{0\} &\iff \lim \tfrac{1}{k} \log \|T^{-a_k} x\| \le 0 \ ,
\end{aligned}$$

where $\|\cdot\|$ is a certain ($\equiv$ any) norm in $\mathfrak{N}$.

(iii) *The subspaces $\mathfrak{N}_-, \mathfrak{N}_-^*, \mathfrak{N}_0, \mathfrak{N}_+, \mathfrak{N}_+^*$ are T-invariant Lie subalgebras of $\mathfrak{N}$.*

(iiii) *Let $\mathcal{N}_-, \mathcal{N}_-^*, \mathcal{N}_0, \mathcal{N}_+, \mathcal{N}_+^*$ be the simply connected subgroups of $\mathcal{N}$ corresponding to the subalgebras $\mathfrak{N}_-, \mathfrak{N}_-^*, \mathfrak{N}_0, \mathfrak{N}_+, \mathfrak{N}_+^*$, respectively, and identified with the corresponding subalgebras by the Baker–Campbell–Hausdorff formula. Then all these groups are T-invariant, and any element $n \in \mathfrak{N}$ can be uniquely decomposed as*

$$n = n_- \times n_0 \times n_+ \ , \qquad n_- \in \mathfrak{N}_-, n_0 \in \mathfrak{N}_0, n_+ \in \mathfrak{N}_+ \ . \tag{3.5.1}$$

The map $n \mapsto (n_-, n_0, n_+)$, $\mathfrak{N} \to \mathfrak{N}_- \times \mathfrak{N}_0 \times \mathfrak{N}_+$ is polynomial and linear in principal terms.

Proof. Since the weight subsets $\Lambda_-, \Lambda_0, \Lambda_+$ are invariant with respect to the complex conjugation, (i) and (ii) are obvious. Further, the description (ii) implies (iii). Finally, since T preserves the lower central series filtration $\{\mathfrak{N}_l\}$, property (i) applied to any $\mathfrak{N}_l$ implies that

$$\mathfrak{N}_l = (\mathfrak{N}_l \cap \mathfrak{N}_-) \oplus (\mathfrak{N}_l \cap \mathfrak{N}_0) \oplus (\mathfrak{N}_l \cap \mathfrak{N}_+) \ .$$

Thus, the polynomial map $(n_-, n_0, n_+) \mapsto n_- \times n_0 \times n_+$ is linear in principal terms, which implies that it is invertible, and its inverse map is also polynomial and linear in principal terms. □

3.5.5. Let $S = A \leftthreetimes \mathcal{N}$ be the semi-direct product determined by the action T. Denote the corresponding coordinate projections by $\alpha : (a, n) \mapsto a \in A$ and $\Pi : (a, n) \mapsto n$. For a given vector $\mathbf{a} \in \mathbf{A}$ let $\Pi_-(a, n) = \pi_-(n)$, where π_- is the projection from $\mathcal{N}$ to $\mathcal{N}_-$ determined by the decomposition (3.5.1).

Since $S = \mathcal{N}A = \mathcal{N}_-\mathcal{N}_0\mathcal{N}_+A = \mathcal{N}_-\mathcal{N}_+^*A$, the homogeneous space $S/\mathcal{N}_+^*A$ can be identified with $\mathcal{N}_- \cong \mathfrak{N}_-$, and the action of a group element $(a, n) = na \in S$ on $\mathfrak{N}_-$ has the form

$$(a, n).x = na.x = \pi_-(n \times T^a x) \ . \tag{3.5.2}$$

In particular, $(a, n).\mathbf{0} = \pi_-(n)$, where $\mathbf{0}$ is the zero vector in $\mathfrak{N}$, and $a.x = \pi_-(T^a x) = T^a x$, because the algebra $\mathfrak{N}_-$ is T-invariant (Proposition 3.5.4).

3.5.6. By [Ka91, Lemma 2.3], if μ is a probability measure with a finite first moment on a finitely generated group G, and $G' \subset G$ – a normal subgroup of finite index (so that it is also finitely generated), then there exists a probability measure μ' on G' with finite first moment in G' such that the Poisson boundaries $\Gamma(G,\mu)$ and $\Gamma(G',\mu')$ are isomorphic. Thus, by Proposition 3.5.2 the problem of describing the Poisson boundary of a probability measure with a finite first moment on a polycyclic group G is reduced to considering only the case when G is splittable.

Let now μ be a probability measure with a finite first moment on a splittable polycyclic group G. Being splittable, G is contained in an $\mathcal{S}$-group, so that we may assume without loss of generality that the group G itself is an S-group $G = A \leftthreetimes N$. Denote by $\mathcal{N} \supset N$ the Lie hull of the group N, and by $\mathfrak{N}$ the Lie algebra of the group $\mathcal{N}$. We keep the notations from **3.5.4**, **3.5.5**.

Theorem. *Let μ be a probability measure with a finite first moment on an $\mathcal{S}$-group $G = A \leftthreetimes N$. Denote by μ_A the projection of the measure μ onto A, and by $\overline{\mu}_A = \sum \mu_A(a)a \in \mathbf{A}$ the barycenter of the measure μ_A. Let $\mathcal{N} = \mathcal{N}_-(\overline{\mu}_A)\mathcal{N}_+^*(\overline{\mu}_A)$, $n = n_- n_+^*$ be the decomposition (3.5.1) of the group $\mathcal{N}$ determined by the vector $\overline{\mu}_A$, and $\Pi_- : (a,n) \mapsto n_-$ be the corresponding map from G to the G-space $S/\mathcal{N}_+^*(\overline{\mu}_A)A \cong \mathfrak{N}_-(\overline{\mu}_A)$. Then for $\mathbf{P}$-a.e. sample path $\{x_k\}$ of the random walk (G,μ) there exists the limit*

$$\lim_{k\to\infty} \Pi_-(x_k) \in \mathfrak{N}_-(\overline{\mu}_A) ,$$

and $\mathfrak{N}_-(\overline{\mu}_A)$ with the corresponding limit measure coincides with the Poisson boundary of the pair (G,μ).

We shall need a couple of auxiliary statements before giving a proof.

3.5.7. Denote by $\mathcal{P}$ the vector space of complex polynomials on $\mathfrak{N}_-$. For any $P \in \mathcal{P}$, $g \in S$ let $P.g(x) = P(g.x)$. One can easily verify (see also [Ra77, Lemme 3.5]) that

1) If $a \in A$, then $P.a \in \mathcal{P}$, and $\deg P.a = \deg P$ (because the action T in $\mathfrak{N}$ preserves the filtration $\{\mathfrak{N}_l\}$);
2) If $n \in \mathcal{N}$, then $P.n \in \mathcal{P}$ with $\deg P.n = \deg P$, and $\deg(P - P.n) < \deg P$ (because the multiplication $\times$ in $\mathfrak{N}$ and the decomposition $n = n_- \times n_0 \times n_+$ are linear in principal terms).

Thus, for any integer l the group S acts by linear transformations on the finite dimensional space $\mathcal{P}_l$ of polynomials of nilpotent degree $\le l$. For our purposes it is sufficient to consider only the space $\mathcal{P}_r$, where r is the nilpotency class of $\mathfrak{N}$. As it follows from Proposition 3.5.4 and its proof, any basis $\{e_i\}$ in $\mathfrak{N}_-^{\mathbb{C}}$ consisting of weight vectors of the action T is adapted to the lower central series filtration of $\mathfrak{N}_-$ (cf. **3.5.3**). Denote by $\lambda_i \in \Lambda$ the weight of the vector e_i, and by $\varphi_i \in \mathcal{P}_r$ the corresponding coordinate function (so that $\deg \varphi_i = \deg e_i$). Monomials $\varphi^l = \prod \varphi_i^{l_i}$ corresponding to multi-indices $l = (l_i)$, $l_i \ge 0$ with $\deg \varphi^l = \sum l_i \deg \varphi_i \le r$ constitute then a basis in $\mathcal{P}_r$, and this basis contains all coordinate functions φ_i. We shall order these monomials according to their degree, so that the zero degree monomial $\mathbf{1} = \varphi^0$ comes the first, and then any monomial of a lower degree always comes before all monomials of a higher degree.

Denote by $M(g)$, $g \in S$ the matrix of the transformation $P \mapsto P.g$ in this basis. Then $M(g_1g_2) = M(g_2)M(g_1)$, so that $g \mapsto M(g)$ is an antirepresentation of the group S in the vector space $\mathcal{P}_r$. The matrices $M(a), a \in A$ are diagonal with entries $\lambda^l(a) =$

$\prod \lambda_i^{l_i}(a)$ as it follows from formula (3.5.2). In particular, the entry at the top of the diagonal corresponding to the zero multi-index $(0,\dots,0)$ is always 1. The matrices $M(n)$, $n \in \mathcal{N}$ are upper triangular with 1's on the diagonal because of the property (2) above, and formula (3.5.2) implies that the first row of the matrix $M(n)$ consists of the entries $\prod \varphi_i^{l_i}(\pi_-(n))$. Thus, we have

Proposition. *For an arbitrary element $g = (a,n) = na \in S$ the matrix $M(g) = M(a)M(n)$ of the action $P \mapsto P.g$ in the space $\mathcal{P}_r$ has the form*

$$M(g) = \begin{pmatrix} 1 & m(g) \\ 0 & M'(g) \end{pmatrix} ,$$

where $m(g) = m(\pi_-(n))$ is the $(\dim \mathcal{P}_r - 1)$-dimensional vector with components $\varphi^l(\pi_-(n))$ (where $1 \le \sum l_i \deg e_i \le r$), and the $(\dim \mathbf{P}_r - 1) \times (\dim \mathbf{P}_r - 1)$ matrix $M'(g)$ is upper triangular with diagonal entries $\lambda^l(a)$, $1 \le \sum l_i \deg e_i \le r$.

Remark. Analogously, this statement (with obvious notational modifications) is also true for the action of the group $S = A \leftthreetimes \mathcal{N}$ on the homogeneous space $S/\mathcal{N}_+ A \cong \mathfrak{N}_-^*$.

3.5.8. Below for estimating norms of the matrices $M(g)$ we shall also need the following elementary result.

Proposition [Ra77, Lemme 9.4]**.** *Let (Z_k) be a sequence of non-degenerate upper triangular matrices of the same order d. If $\limsup \log \|Z_{k+1} Z_k^{-1}\|/k \le 0$, and for diagonal elements Z_k^{ii}, $1 \le i \le d$ there exist limits $z^i = \lim \log |Z_k^{ii}|/k$, then $\lim \log \|Z_k\|/k = \max z^i$.*

3.5.9. Proof of Theorem 3.5.6.

I. Convergence. Since the measure μ has a finite first moment in G, its projection $\mu_A = \alpha(\mu)$ onto A also has a finite first moment, and $\sum \log \|M(g)\| \mu(g) < \infty$, where $M(g)$ are the matrices from Proposition 3.5.7. Denote by

$$M_k = M(a_k, n_k) = \begin{pmatrix} 1 & m_k \\ 0 & M'_k \end{pmatrix} ,$$

the matrices corresponding to the increments (a_k, n_k) of the random walk. Then the product matrices $Z_k = M'_k M'_{k-1} \dots M'_1$ satisfy conditions of Proposition 3.5.8 with diagonal limits $\sum l_i \log \lambda_i(\mathbf{a}) < 0$, so that $\lim \log \|Z_k\|/k < 0$.

Since $x_k = (a_1, n_1) \dots (a_k, n_k)$, the matrix

$$M(x_k) = \begin{pmatrix} 1 & m'(x_k) \\ 0 & M'(x_k) \end{pmatrix} = M_k \cdots M_1$$

has the entries $M'(x_k) = Z_k$ and $m(x_k) = m_1 + m_2 Z_1 + \dots + m_k Z_{k-1}$. As $\log \|m_k\| \le \log \|M_k\| = o(k)$ by finiteness of the first moment of μ, and $\lim \log \|Z_k\|/k < 0$, the sequence of vectors $m(x_k)$ converges a.e., which implies a.e. convergence of the sequence $\Pi_-(x_k)$, because the vector $m(g) = m(\Pi_-(g))$ contains all coordinates of $\Pi_-(g)$.

II. Maximality. We shall use Theorem 2.4.5. For the reflected measure $\check\mu$ its projection onto A is the reflected measure of the projection μ_A. Thus, $\overline{\check\mu_A} = -\overline{\mu}_A$. By definition, $\mathcal{N}_-(\mathbf{a}) = \mathcal{N}_+(-\mathbf{a}), \mathcal{N}_0(\mathbf{a}) = \mathcal{N}_0(-\mathbf{a})\ \forall\, \mathbf{a} \in \mathbf{A}$, so that by the first part of the proof applied to the measure $\check\mu$ the homogeneous space $\mathcal{N}_+(\overline{\mu}_A) = S/Ng_-^*(\overline{\mu}_A)A$ is a $\check\mu$-boundary.

Now we have to construct G-equivariant strips $S(n_-, n_+) \subset G,\ n_- \in \mathfrak{N}_-, n_+ \in \mathfrak{N}_+$. Decomposition (3.5.1) implies that for any $n_- \in \mathfrak{N}_-, n_+ \in \mathfrak{N}_+$

$$(3.5.3)\quad n_-\mathcal{N}_+^* \cap n_+\mathcal{N}_-^* = n_+(n_+^{-1}n_-\mathcal{N}_+^* \cap \mathcal{N}_-^*) = n_+(n'_-\mathcal{N}_+^* \cap \mathcal{N}_-^*) = n_+n'_-\mathcal{N}_0 = \widetilde{n}\mathcal{N}_0\ ,$$

where $n'_- = n_+^{-1}.n_- \in \mathfrak{N}_-$ (recall that by "dot" we denote the action (3.5.2) on $\mathfrak{N}_-$), and $\widetilde{n} = n_+n'_-$. So, the intersection of any two $\mathcal{N}_+^*$ and $\mathcal{N}_-^*$ cosets in $\mathcal{N}$ is a $\mathcal{N}_0$-coset (one can easily see that in fact any $\mathcal{N}_0$-coset can be uniquely presented in this way).

The group N is cocompact in $\mathcal{N}$, so that there exists a compact set $K \subset \mathcal{N}$ such that for any translation $nK,\ n \in \mathcal{N}$ the set of N-points $nK \cap N$ in nK is non-empty. Then

$$(3.5.4)\qquad S(n_-, n_+) = \big[(n_-\mathcal{N}_+^* \cap n_+\mathcal{N}_-^*)K \cap N\big]A \subset G$$

is a G-equivariant map assigning to pairs of points from $\mathfrak{N}_- \times \mathfrak{N}_+$ non-empty subsets of G. Clearly, S is measurable, and, as it follows from (3.5.3), all strips have the form

$$S(n_-, n_+) = (\widetilde{n}\mathcal{N}_0K \cap N)A = S(\widetilde{n})$$

for a certain $\widetilde{n} = \widetilde{n}(n_-, n_+) \in \mathcal{N}$.

Now fix linear norms in $\mathbf{A} \supset A$ and in $\mathfrak{N}$ and let

$$\mathcal{G}_k = \{(a, n) \in G : \|a\|, \|\pi_0(n)\| \le e^k\}\ .$$

We shall verify that the strips (3.5.4) and the gauge $\mathcal{G} = (\mathcal{G}_k)$ satisfy conditions of Theorem 2.4.5.

First note that any strip $S(\widetilde{n})$ has at most exponential growth with respect to the gauge $(\mathcal{G}_k)$ (although the gauge sets $\mathcal{G}_k$ are themselves infinite). Indeed, let

$$g = (a, n) = na = n_-n_0n_+a \in n\mathcal{N}_0KA \cap \mathcal{G}_k\ ,$$

i.e., $\|a\|, \|n_0\| \le e^k$ and $n_-n_0n_+ \in \widetilde{n}\mathcal{N}_0K$. The latter formula means that there exists $n' \in K$ such that $\widetilde{n}^{-1}n_-n_0n_+{n'}^{-1} \in \mathcal{N}_0$. On the other hand, since the group multiplication in $\mathcal{N} \cong \mathfrak{N}$ is polynomial and linear in principal terms, for any $n_0 \in \mathcal{N}_0$ and $n_1, n_2 \in \mathcal{N}$ there exist uniquely determined $n_- \in \mathcal{N}_-$ and $n_+ \in \mathcal{N}_+$ such that the product $n_1n_-n_0n_+n_2$ belongs to $\mathcal{N}_0$, and the map $\varphi : (n_0, n_1, n_2) \mapsto n_-n_0n_+$ is polynomial. The set K is compact, and $\widetilde{n}$ is fixed, so that there is a constant $C = C(\widetilde{n}, K)$ such that $\|\varphi(n_0, \widetilde{n}^{-1}, {n'}^{-1})\| \le C\|n_0\|^r$ for any $n' \in K$. Thus, $\|n\| \le Ce^{kr}$. Since the groups $\mathbf{A}$

and $\mathcal{N}$ have polynomial growth and the embeddings $A \subset \mathbf{A}$, $N \subset \mathcal{N}$ are discrete, we conclude that for any $\widetilde{n} \in \mathcal{N}$

$$\limsup_{k\to\infty} \frac{1}{k} \log \operatorname{card} \left[S(\widetilde{n}) \cap \mathcal{G}_k \right] < \infty \,. \tag{3.5.5}$$

By using the same argument as in the first part of this proof and considering the G-action on the space of polynomials on $\mathfrak{N}_-^*$ (see Remark after Proposition 3.5.7), one shows that a.e. $\log \|\Pi_0(x_k)\| = o(k)$. The A-component $\alpha(x_k)$ of x_k performs the random walk on A determined by the measure μ_A with a finite first moment, so that a.e. $\|\alpha(y_k)\|/k \to \|\overline{\mu}_A\| < \infty$, whence a.e. $|x_k|_{\mathcal{G}} = o(k)$. In combination with (3.5.5) it means that the conditions of the Theorem 2.4.5 are satisfied.

Corollary 1. *If μ is a symmetric measure with a finite first moment on a polycyclic group G, then the Poisson boundary $\Gamma(G, \mu)$ is trivial.*

Corollary 2. *If the Poisson boundary is non-trivial for a certain symmetric probability measure μ with a finite first moment on a finitely generated solvable group G, then G contains an infinitely generated subgroup.*

3.5.10. *Remarks.* 1. The proof of convergence in Theorem 3.5.6 is similar to the proof of an analogous statement in the setup of real Lie groups [Ra77].

2. Another way of obtaining a description of the Poisson boundary of a polycyclic group G consists in embedding G into the matrix group $GL(d, \mathbb{Z})$ and using the description of the Poisson boundary for this group, see Section 3.4. In this approach the Poisson boundary is identified with (a subset of) a certain flag space in $\mathbb{R}^d$. One could also use the *global law of large numbers* for solvable Lie groups, which allows one to approximate (in the enveloping solvable Lie group $\widetilde{G}$) a.e. sample path $\boldsymbol{x} = \{x_n\}$ by the sequence of powers g^n of a certain group element $g = g(\boldsymbol{x}) \in \widetilde{G}$ [Ka91].

3. The automorphisms $T^a : \mathfrak{N} \to \mathfrak{N}$, $a \in A$ preserve the cocompact group N, hence $|\det T^a| \equiv 1$, and the subalgebras $\mathfrak{N}_-(\mathbf{a})$ and $\mathfrak{N}_+(\mathbf{a})$ are trivial or non-trivial simultaneously in perfect keeping with the fact that the Poisson boundaries $\Gamma(G, \mu)$ and $\Gamma(G, \check{\mu})$ are trivial or non-trivial simultaneously (which follows from Theorem 1.6.7).

4. The proof of maximality in Theorem 3.5.6 in a sense is a combination of proofs in two important particular cases when the neutral subgroup $\mathcal{N}_0$ is either trivial or coincides with the whole group $\mathcal{N}$. In the first case the strips in G have the form $S(\widetilde{n}) = (\widetilde{n}K \cap N)A$, and the proof of maximality becomes trivial (modulo Theorem 2.4.5) – cf. below Theorem 3.6.2. In the second case, if $\overline{\mu}_A = 0$ (in particular, if the measure μ is symmetric) Theorem 3.5.6 reduces to showing that the Poisson boundary of the measure μ is trivial. This can be done by a direct estimate of the rate of escape of the random walk (G, μ). If (a_k, n_k) are the increments of the random walk, then its position at time k is

$$x_k = (a_1 + \dots a_k, n_1 \times T^{a_1} n_2 \times \dots T^{a_1 + \dots a_{k-1}} n_k) \,.$$

If $|\cdot|$ is a word length on N, then $\log^+ \|n_k\| = o(k)$ (provided the measure μ has a finite first moment). Since $a_1 + \dots + a_k = o(k)$, it implies that $\log^+ \|T^{a_1 + \dots a_{k-1}} n_k\| = o(k)$, so that $\log^+ \|\pi(y_k)\| = o(k)$. Thus, the entropy of the random walk is zero, because the nilpotent group G has polynomial growth.

5. As one could expect, the boundary theory for polycyclic groups is parallel to that for solvable Lie groups (although the methods are quite different). The description of the Poisson boundary for polycyclic groups obtained in Theorem 3.5.6 is essentially the same as for solvable Lie groups [Az70], [Ra77].

3.6. Semi-direct and wreath products.

3.6.1. Let $G = A \leftthreetimes H$ be the *semi-direct product* determined by an action T of a group A by automorphisms of another group H, i.e., the group operation in G is $(a_1, h_1)(a_2, h_2) = (a_1 a_2, h_1 \cdot T^{a_1} h_2)$. We assume that the groups A and H are embedded into G by the maps $a \mapsto (a, e_H)$ and $h \mapsto (e_A, h)$. The following is obvious:

Lemma. *Let $G = A \leftthreetimes H$ be a semi-direct product, X is a G-space, and $\pi : X \to H$ is an H-equivariant map. Then the map*

$$S : x \mapsto \left\{a\pi(a^{-1}x)\right\}_{a\in A} = \left\{\left(a, T^a\pi(a^{-1}x)\right) : a \in A\right\} \subset G$$

is G-equivariant.

3.6.2. Lemma 3.6.1 in combination with Theorem 2.4.6 then immediately implies

Theorem. *Let G be a finitely generated group decomposable as a semi-direct product $A \leftthreetimes H$, and let μ be a probability measure on G. Suppose that (B_-, λ_-) and (B_+, λ_+) are $\check{\mu}$- and μ-boundaries, respectively, and there exists a measurable H-equivariant map $\pi : B_- \times B_+ \to H$. If either*

(a) *The measure μ has a finite first moment, and the growth of A is subexponential;*

or

(b) *The measure μ has a finite first logarithmic moment and a finite entropy, and the growth of A is polynomial;*

then the boundaries (B_-, λ_-) and (B_+, λ_+) are maximal.

Remark. The class of finitely generated groups of polynomial growth coincides with the class of virtually nilpotent finitely generated groups [Gr81]. On the other hand, there exist examples of groups whose growth is intermediate between polynomial and exponential [Gi85].

3.6.3. For an integer $p > 1$ let $BS(1,p)$ be the *Baumslag–Solitar group* determined by two generators a, b and the relation $aba^{-1} = b^p$ [BS62]. The group $BS(1,p)$ coincides with the affine group of the ring $\mathbb{Z}[\frac{1}{p}] = \{k/p^l : k \in \mathbb{Z}, l \in \mathbb{Z}_+\}$ and can be presented as the group of matrices

$$(z, f) = \begin{pmatrix} p^z & f \\ 0 & 1 \end{pmatrix}, \qquad f = \frac{k}{p^l},$$

where $a = (1,0)$ and $b = (0,1)$, so that $BS(1,p)$ is isomorphic to the semi-direct product $\mathbb{Z} \leftthreetimes \mathbb{Z}[\frac{1}{p}]$ determined by the action $T^z f = p^z f$. The group $BS(1,p)$ is solvable of degree 2 and has exponential growth.

For a number $f \in \mathbb{Z}[\frac{1}{p}] \setminus \{0\}$ let $\|f\| = 1 + \log^+ |f| + \log^+ |f|_p$, where $|f|$ is the ordinary absolute value of f and $|f|_p = \min\{p^k : p^k f \in \mathbb{Z}\}$ (so that if p is a prime then $|f|_p$ is the p-adic absolute value of f), and put $\|0\| = 0$. One can easily show that the gauge $(x,f) \mapsto |x| + \|f\|$ is equivalent to a word gauge in $BS(1,p)$ in the sense of **2.2.1**. Denote by $\mathbb{Q}_p$ the completion of the ring $Z[\frac{1}{p}]$ with respect to the the distance $|f_1 - f_2|_p$. If p is a prime, then $\mathbb{Q}_p$ is the field of p-adic numbers. In a natural way the Cantor set $\mathbb{Q}_p$ and the real line $\mathbb{R}$ (which is the completion of $Z[\frac{1}{p}]$ in the usual metric) can be considered as two boundaries of the group $BS(1,p)$ ("upper" and "lower"), see [KV83], [FM97].

3.6.4. Theorem. *Let μ be a probability measure on the group $G = BS(1,p)$ with a finite first moment and such that the group $\operatorname{gr}(\mu)$ is non-abelian. Denote by $\overline{\mu}_{\mathbb{Z}}$ the mean of the projection of the measure μ onto $\mathbb{Z}$ determined by the homomorphism $BS(1,p) \to \mathbb{Z}$, $(x,f) \mapsto x$.*

(i) *If $\overline{\mu}_{\mathbb{Z}} < 0$, then for* **P**-a.e. *path (x_n, φ_n) of the random walk (G,μ) there exists the limit*

$$\lim_{n\to\infty} \varphi_n = f_\infty \in \mathbb{R} \,,$$

and the Poisson boundary of the pair (G,μ) is isomorphic to $\mathbb{R}$ with the resulting limit measure λ;

(ii) *If $\overline{\mu}_{\mathbb{Z}} = 0$, then the Poisson boundary of the pair (G,μ) is trivial;*

(iii) *If $\overline{\mu}_{\mathbb{Z}} > 0$, then for* **P**-a.e. *path (x_n, φ_n) of the random walk (G,μ) there exists the limit*

$$\lim_{n\to\infty} \varphi_n = f_\infty \in \mathbb{Q}_p \,,$$

and the Poisson boundary of the pair (G,μ) is isomorphic to $\mathbb{Q}_p$ with the resulting limit measure λ.

Proof. Let

$$\begin{aligned}\{(x_n,\varphi_n)\} &= (h_1,f_1)(h_2,f_2)\cdots(h_n,f_n)\\ &= (h_1 + h_2 + \dots h_n, f_1 + p^{x_1} f_2 + \dots + p^{x_{n-1}} f_n)\end{aligned}$$

be a path of the random walk (G,μ) with increments (h_i, f_i).

If $\overline{\mu}_{\mathbb{Z}} = 0$, then the random walk $\{x_n\}$ on $\mathbb{Z}$ is recurrent, so that the Poisson boundary of (G,μ) coincides with the Poisson boundary of the induced random walk on the abelian group $\mathbb{Z}[\frac{1}{p}] \subset G$ [Fu71], [Ka91], the latter being trivial (see **1.2.8**). Another proof of boundary triviality in this case can be obtained by showing that the rate of escape of the random walk (G,μ) is zero (see Corollary of Theorem 2.3.2).

Suppose now that $\overline{\mu}_{\mathbb{Z}} \neq 0$. Then **P**-a.e. $x_n/n \to \overline{\mu}_{\mathbb{Z}}$ and $\log^+ |f_n|, \log^+ |f_n|_p = o(n)$ (by the law of large numbers applied to the i.i.d. random variables $\|f_n\|$), which proves convergence in the cases (i) and (iii). Since stabilizers of points from $\mathbb{R}$ and $\mathbb{Q}_p$ are abelian with respect to the affine action of G, the resulting limit measures must be non-trivial.

Now we have to prove maximality of the arising μ-boundaries. In view of Theorem 3.6.2 we shall do it simultaneously for the cases (i) and (iii), because if one of the measures $\mu, \check{\mu}$ has negative drift, then the other one has positive drift.

For any two points $x \in \mathbb{R}$, $\xi \in \mathbb{Q}_p$ let $\pi(x, \xi) = x + \{\{\xi\} - \{x\}\} \in \mathbb{Z}[\frac{1}{p}]$, where $x \mapsto \{x\}$ is the function assigning to a real or p-adic number its fractional part $0 \le \{x\} < 1$. Then, clearly, $\pi(x+t, \xi+t) = t + \pi(x, \xi)$ for any $t \in \mathbb{Z}[\frac{1}{2}]$, so that the conditions of Theorem 3.6.2 are satisfied. □

Remarks. 1. This result is in perfect keeping with the fact that $BS(1,p)$ is a lattice in the product of affine groups of $\mathbb{R}$ and of $\mathbb{Q}_p$. Depending on the sign of $\overline{\mu}_{\mathbb{Z}}$, the random walk then acts contractively either on the real or on the p-adic line. Note that for the real affine group [Az70] (resp., for the p-adic affine group [CKW94]) one gets a non-trivial Poisson boundary isomorphic to $\mathbb{R}$ (resp., to $\mathbb{Q}_p$) only if the random walk in contracting in the real (resp., p-adic) metric.

2. It would be interesting to investigate the Poisson boundary for higher-dimensional solvable groups over $\mathbb{Z}[\frac{1}{p}]$, for example, for the group of triangular matrices. For these groups the Poisson boundary should be mixed – consisting of both real and p-adic components. This problem is also closely related with finding out a description of the Poisson boundary for random walks on Lie groups over $\mathbb{Q}$, in which case the adele groups should come into play.

3.6.5. Denote by $\mathbf{fun}\,(A,B)$ the direct sum of isomorphic copies of a group B indexed by the elements from another group A. The group $\mathbf{fun}\,(A,B)$ can be considered as the group of *finitely supported* A-valued *configurations* on A with the operation of pointwise multiplication, and it is endowed with a natural action of the group A by translations: $T^a f(a') = f(a^{-1}a')$. Below we shall also use the group $\mathbf{Fun}\,(A,B)$ of *all* (not necessarily finitely supported) B-valued configurations on A. The semi-direct product $A \leftthreetimes \mathbf{fun}\,(A,B)$ corresponding to the action T of the group A on $\mathbf{fun}\,(A,B)$ by translations is called the (restricted) *wreath product* of the *active group* A and the *passive group* B [KM79]. Note that the groups $BS(1,p)$ considered above are homomorphic images of the wreath product $\mathbb{Z} \leftthreetimes \mathbf{fun}\,(\mathbb{Z},\mathbb{Z})$ under the maps $(x,f) \mapsto (x, \sum p^k f(k))$.

The wreath product is finitely generated if both its active and passive groups are. Given sets of generators $A_0 \subset A, B_0 \subset B$ the corresponding set of generators of G is the union of the sets $\{(a,\phi) : a \in A_0\}$ and $\{(e_A, \varepsilon_b) : b \in B_0\}$, where ϕ is the identity of $\mathbf{fun}\,(A,B)$ (i.e., $\phi(a) = e_B$ for all $a \in A$), and $\varepsilon_b \in \mathbf{fun}\,(A,B)$ is defined as $\varepsilon_b(e_A) = b$ and $\varepsilon_b(a) = e_B$ otherwise.

3.6.6. Theorem. *Let $G = A \leftthreetimes \mathbf{fun}\,(A,B)$ be a finitely generated wreath product, and μ – a probability measure on G. Suppose that*

(i) *The active group A has subexponential growth;*

(ii) *The measure μ has a finite first moment;*

(iii) *There exists a homomorphism $\psi : A \to \mathbb{Z}$ such that the mean $\overline{\mu}_{\mathbb{Z}}$ of the measure $\mu_{\mathbb{Z}} = \psi(\mu)$ is non-zero.*

Then for **P**-a.e. *sample path* $\{(x_n, \varphi_n)\}$ *the configurations* φ_n *converge pointwise to a limit configuration* $\lim \varphi_n \in \mathbf{Fun}(A, B)$, *and the Poisson boundary of the pair* (G, μ) *is isomorphic to* $\mathbf{Fun}(A, B)$ *with the resulting limit measure* λ.

Proof. I. Convergence. Fix certain word gauges $|\cdot|_A$ and $|\cdot|_B$ on the groups A and B, and denote by $|\cdot|$ the corresponding word gauge on G (as explained at the end of **3.6.5**). For a configuration $f \in \mathbf{fun}(A, B)$ let $\|f\| = \max\{|a|_A : f(a) \neq e_B\}$. Then obviously $\|f\| \leq |(a, f)|$ for any $a \in A$. Positions of the random walk (G, μ) at times $n, n+1$ are connected with the formula

$$(x_{n+1}, \varphi_{n+1}) = (x_n, \varphi_n)(h_{n+1}, f_{n+1}) = (x_n h_{n+1}, \varphi_n T^{x_n} f_{n+1}) \, , \tag{3.6.1}$$

where (h_n, f_n) are the independent μ-distributed increments of the random walk. By condition (ii) **P**-a.e. $\|f_{n+1}\| = o(n)$, whence by (iii) we obtain convergence of the configurations φ_n.

II. Maximality. Suppose, for the sake of concreteness, that $\overline{\mu}_{\mathbb{Z}} > 0$. As it follows from the first part of the proof, the limit measure λ_+ on $\mathbf{Fun}(A, B)$ corresponding to the random walk (G, μ) has the property that the restriction of λ_+-a.e. configuration to the set $A_- = \{a \in A : \psi(a) < 0\}$ is finite. In the same way, for the limit measure λ_- of the reflected random walk $(G, \check{\mu})$ the restriction of λ_--a.e. configuration to the set $A_+ = \{a \in A : \psi(a) \geq 0\}$ is also finite. Thus, for $\lambda_- \otimes \lambda_+$-a.e. pair of configurations $\Phi_-, \Phi_+ \in \mathbf{Fun}(A, B)$ the configuration $f = \pi(\Phi_-, \Phi_)$ defined as

$$f(a) = \begin{cases} \Phi_-(a) \, , & a \in A_+ \\ \Phi_+(a) \, , & a \in A_- \end{cases}$$

belongs to $\mathbf{fun}(A, B)$, and obviously the map π is $\mathbf{fun}(A, B)$-equivariant, so that the claim follows from Theorem 3.6.2 (a). □

3.6.7. Example. Let $A = \mathbb{Z}^k$ and $B = \mathbb{Z}_2 = \{0, 1\}$. The corresponding wreath products $G_k = \mathbb{Z}^k \leftthreetimes \mathbf{fun}(\mathbb{Z}_k, \mathbb{Z}_2)$ were first considered in [KV83] as a source of several examples and counterexamples illustrating the relationship between growth, amenability and the Poisson boundary for random walks on groups. Let μ_0 be a probability measure on $\mathbb{Z}^k$, and $\mu(x, \varepsilon) = \mu_0(x)$ be its lift to G_k, where $\varepsilon_1 \in \mathbf{fun}(\mathbb{Z}^k, \mathbb{Z}_2)$ is the configuration taking the value 1 at the identity of $\mathbb{Z}^k$ and the value 0 otherwise. Then in view of formula (3.6.1) the random walk $\{(x_n, \varphi_n)\}$ on G_k governed by the measure μ has the following interpretation: its projection $\{x_n\}$ is the random walk on $\mathbb{Z}^k$ governed by the measure μ_0, whereas the configuration component φ_{n+1} is obtained from φ_n just by changing its value at the point x_n. One can think that there is a lamp at each point of $\mathbb{Z}^k$, and a lamplighter performs the random walk governed by the measure μ_0 on $\mathbb{Z}_k$ flipping the light at all points through which he passes (because of this description the groups G_k are sometimes referred to as *groups of dynamical configurations* [KV83] or *lamplighter groups* [LPP96]).

The Poisson boundary of the random walk (G_k, μ) is non-trivial iff the random walk $(\mathbb{Z}^k, \mu_0)$ is transient. Indeed, if $(\mathbb{Z}^k, \mu_0)$ is recurrent, then the Poisson boundary of (G_k, μ) coincides with the (trivial) Poisson boundary of the induced random walk on the abelian group $\mathbf{fun}(\mathbb{Z}^k, \mathbb{Z}_2)$ (cf. the proof of Theorem 3.6.4 (ii)). On the other hand, if $(\mathbb{Z}^k, \mu_0)$ is transient, then a.e. $x_n \to \infty$, so that the configurations φ_n pointwise stabilize and provide a non-trivial behaviour at infinity. Theorem 3.6.6 implies that if the measure μ_0 has a finite first moment, and its mean is non-zero, then the Poisson boundary of (G_k,μ) is the space of limit configurations from $\mathbf{Fun}(\mathbb{Z}^k,\mathbb{Z}_2)$. Whether this is *always* true when the quotient random walk $(\mathbb{Z}^k,\mu_0)$ is transient, in particular, if μ_0 has a finite first moment with zero mean, is an open question.

3.6.8. Let $\{x_n\}_{n=0}^\infty$ be a arbitrary homogeneous Markov chain on a countable state space X with transition probabilities $p(x,y),\ x,y\in X$. Denote by $X^{\mathbb{Z}_+}$ the space of (unilateral) paths $\{x_n\}_{n=0}^\infty$ in X, and by $\mathbf{P}_\theta$ the probability measure on $X^{\mathbb{Z}_+}$ determined by an initial distribution θ.

The group $S(\infty)$ of *finite permutations* of the parameter set $\mathbb{Z}_+=\{0,1,2,\dots\}$ acts on the path space $X^{\mathbb{Z}_+}$. Denote by σ the measurable partition of the path space $(X^{\mathbb{Z}_+},\mathbf{P}_\theta)$ which is the envelope of the trajectory equivalence relation of this action. The corresponding σ-algebra $\mathcal{S}$ in the path space, i.e., the (completed) σ-algebra of $S(\infty)$-invariant sets, is called the *exchangeable* (or: *symmetric*) σ-algebra of the chain $\{x_n\}$. Let us say that that the quotient space $(X^{\mathbb{Z}_+},\mathbf{P}_\theta)/\sigma$ is the *exchangeable boundary* of the chain $\{x_n\}$.

We introduce the *extended chain* $\{(x_n,\sum_{k=0}^{n-1}\delta_{x_k})\}$ on the state space $X\times\mathbf{fun}(X,\mathbb{Z})$, where $\mathbf{fun}(X,\mathbb{Z})$ is the additive group of finitely supported $\mathbb{Z}$ valued configurations on X. In other words, we add to the states x_n of the original chain the *occupation functions* $\varphi_n=\sum_{k=0}^{n-1}\delta_{x_k}$ saying how many times each of the points of the state space X was visited by the path $\{x_n\}$ up to the time n. The transition probabilities of the extended chain are

$$\widetilde{p}\big((x,f),(y,f+\delta_x)\big)=p(x,y)\,.$$

Clearly, the path space $(X^{\mathbb{Z}_+},\mathbf{P}_\theta)$ of the original chain is isomorphic to the path space of the extended chain with the initial distribution $\theta\otimes\delta_\phi$, where ϕ is the zero configuration.

3.6.9. Lemma [Ka91]. *For an arbitrary initial distribution θ on X the tail and the Poisson boundaries of the extended chain, and the exchangeable boundary of the original chain $\{x_n\}$ all coincide* $\mathbf{P}_\theta-\mathrm{mod}\ 0$.

Proof. Recall that the tail equivalence relation of the extended chain is generated by the synchronous equivalence relation of the shift in its path space (see **1.5.1**):

$$\{(x_n,\varphi_n)\}\approx\{(x',\varphi'\}\iff\exists\,n\ge 0: x_i=x_i',\ \varphi_i=\varphi_i'\ \forall\,i\ge n\ .$$

Since the occupation functions φ_n for the extended chain has the form $\varphi_n=\sum_{k=0}^{n-1}\delta_{x_k}$, we immediately get that the equivalence relation $\approx$ coincides with the trajectory equivalence relation of the group $S(\infty)$ acting on the path space by coordinate permutations, so that the tail boundary of the extended chain coincides with the exchangeable boundary of the original one. Moreover, since the sum of values of φ_n is always n for $\mathbf{P}_\theta$-a.e. sample path $\{(x_n,\varphi_n)\}$, the synchronous and asynchronous equivalence relations of the shift in the path space of the extended chain are the same $\mathbf{P}_\theta-\mathrm{mod}\ 0$, so that the tail and the Poisson boundaries of the extended chain are also the same. □

3.6.10. The exchangeable boundary is trivial $\mathbf{P}_x-\mathrm{mod}\ 0$ for any recurrent state $x\in X$ of the chain $\{x_n\}$. Indeed, recurrence of the state x means recurrence of the set

$\{x\} \times \mathbf{fun}(X, \mathbb{Z})$ for the extended chain. Thus, the Poisson boundary of the extended chain coincides with the (trivial) Poisson boundary of the induced random walk on the abelian group $\mathbf{fun}(X, \mathbb{Z})$ (cf. the proof of Theorem 3.6.4 (ii)). If the chain $\{x_n\}$ satisfies a natural connectivity type condition, then its exchangeable boundary is trivial $\mathbf{P}_\theta$– mod 0 for *any* initial distribution θ [BF64].

On the contrary, transience of the chain $\{x_n\}$ means that any point of the state space is visited by almost all sample paths a finite number of times only. Thereby, the occupation functions φ_n a.e. converge pointwise to a (finite) *final occupation function* φ_∞ (depending on the path $\{x_n\}$). The value $\varphi_\infty(x)$ is the number of times when a point x was visited by the trajectory $\{x_n\}$. Clearly, the final occupation function φ_∞ is measurable with respect to the exchangeable σ-algebra of the chain $\{x_n\}$. When is the exchangeable σ-algebra of a transient chain generated by the final occupation functions? In other words, when does the Poisson boundary of the extended chain coincide with the space of final occupation times? In the case when the chain $\{x_n\}$ is a random walk on a group $G = X$ governed by a measure μ on G, this question by Lemma 3.6.9 can be reformulated as the problem of identifying the Poisson boundary of the random walk on the wreath product $A \leftthreetimes \mathbf{fun}(G, \mathbb{Z})$ governed by the measure $\overline{\mu}(x, \varepsilon) = \mu(x)$, where $\varepsilon_1 \in \mathbf{fun}(G, \mathbb{Z})$ is the configuration taking the value 1 at the identity of G and the value 0 otherwise (cf. **3.6.7**), and by virtue of Theorem 3.6.6 we obtain

Theorem. *Let μ be a probability measure with a finite first moment on a finitely generated group G of subexponential growth. If there exists a homomorphism $\psi : G \to \mathbb{Z}$ such that the mean $\overline{\mu}_{\mathbb{Z}}$ of the measure $\mu_{\mathbb{Z}} = \psi(\mu)$ is non-zero, then the exchangeable boundary of the random walk (G, μ) is isomorphic to the space of final occupation functions.*

Remarks. 1. Since final occupation functions are invariant with respect to the bigger group $\overline{\mathcal{S}}$ of *all* permutations of the index set $\mathbb{Z}_+$, coincidence of the exchangeable boundary with the space of final occupation times implies that any $\mathcal{S}$-invariant subset of the path space is automatically also $\overline{\mathcal{S}}$-invariant (mod 0).

2. The only other result on the description of the exchangeable boundary of a transient Markov chain known to the author is its identification with the space of final occupation functions for transient random walks on $\mathbb{Z}^d$ with a finitely supported measure μ (and also for some other random walks on groups of polynomial growth) by entirely different methods in [JP96].

References

[Al87] G. Alexopoulos, *On the mean distance of random walks on groups*, Bull. Sci. Math. **111** (1987), 189–199.

[An87] A. Ancona, *Negatively curved manifolds, elliptic operators and the Martin boundary*, Ann. of Math. **125** (1987), 495–536.

[An90] A. Ancona, *Théorie du potentiel sur les graphes et les variétés*, Springer Lecture Notes in Math. **1427** (1990), 4–112.

[AS85] M. T. Anderson, R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature*, Ann. of Math. **121** (1985), 429–461.

[Av72] A. Avez, *Entropie des groupes de type fini*, C. R. Acad. Sci. Paris, Sér. A **275** (1972), 1363–1366.

[Av76] A. Avez, *Harmonic functions on groups*, Differential Geometry and Relativity, Reidel, Dordrecht-Holland, 1976, pp. 27–32.

[Az70] R. Azencott, *Espaces de Poisson des groupes localements compacts*, Springer Lecture Notes in Math., vol. 148, Springer, Berlin, 1970.

[Ba89] W. Ballmann, *On the Dirichlet problem at infinity for manifolds of nonpositive curvature*, Forum Math. **1** (1989), 201–213.

[Ba95] W. Ballmann, *Lectures on spaces of nonpositive curvature*, DMV Seminar, vol. 25, Birkhäuser, Basel, 1995.

[BF64] D. Blackwell, D. Freedman, *The tail σ-field of a Markov chain and a theorem of Orey*, Ann. Math. Stat. **35** (1964), 1291–1295.

[Bi91] C. J. Bishop, *A characterization of Poissonian domains*, Ark. Mat. **29** (1991), 1–24.

[Bl55] D. Blackwell, *On transient Markov processes with a countable number of states and stationary transition probabilities*, Ann. Math. Stat. **26** (1955), 654–658.

[BL94] W. Ballmann, F. Ledrappier, *The Poisson boundary for rank one manifolds and their cocompact lattices*, Forum Math. **6** (1994), 301–313.

[BL96] W. Ballmann, F. Ledrappier, *Discretization of positive harmonic functions on Riemannian manifolds and Martin boundary*, Actes de la Table Ronde de Géometrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 77–92.

[Bo69] A. Borel, *Linear Algebraic Groups*, Benjamin, New York, 1969.

[Br71] M. Brelot, *On topologies and boundaries in potential theory*, Springer Lecture Notes in Math., vol. 175, Springer, Berlin, 1971.

[BS62] G. Baumslag, D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201.

[CD60] G. Choquet, J. Deny, *Sur l'equation de convolution $\mu = \mu * \sigma$*, C. R. Ac. Sci. Paris, Sér. A **250** (1960), 799–801.

[CDP90] M. Coornaert, T. Delzant, A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes in Math., vol. 1441, Springer, Berlin, 1990.

[Ce82] N. N. Čencov, *Statistical Decision Rules and Optimal Inference*, Amer. Math. Soc., Providence, R.I., 1982.

[CFS82] I. P. Cornfeld, S. V. Fomin, Ya. G. Sinai, *Ergodic theory*, Springer, New York, 1982.

[CKW94] D. I. Cartwright, V. A. Kaimanovich, W. Woess, *Random walks on the affine group of a homogeneous tree*, Ann. Inst. Fourier (Grenoble) **44** (1994), 1243–1288.

[CS89] D. I. Cartwright, P. M. Soardi, *Convergence to ends for random walks on the automorphism group of a tree*, Proc. Amer. Math. Soc. **107** (1989), 817–823.

[CSW93] D. I. Cartwright, P. M. Soardi, W. Woess, *Martin and end compactifications for nonlocally finite graphs*, Trans. Amer. Math. Soc. **338** (1993), 679–693.

[De75] Y. Derriennic, *Marche aléatoire sur le groupe libre et frontière de Martin*, Z. Wahrscheinlichkeitsth. Verw. Geb. **32** (1975), 261–276.

[De76] Y. Derriennic, *Lois "zéro ou deux" pour les processus de Markov, applications aux marches aléatoires*, Ann. Inst. H. Poincaré Séct. B **12** (1976), 111–129.

[De80] Y. Derriennic, *Quelques applications du théorème ergodique sous-additif*, Astérisque **74** (1980), 183–201.

[De86] Y. Derriennic, *Entropie, théorèmes limites et marches aléatoires*, Springer Lecture Notes in Math. **1210** (1986), 241–284.

[DE90] D. P. Dokken, R. Ellis, *The Poisson flow associated with a measure*, Pacific J. Math **141** (1990), 79–103.

[DM61] E. B. Dynkin, M. B. Malyutov, *Random walks on groups with a finite number of generators*, Soviet Math. Dokl. **2** (1961), 399–402.

[Dy82] E. B. Dynkin, *Markov Processes and Related Problems of Analysis*, London Math. Soc. Lecture Note Series, vol. 54, Cambridge Univ. Press, Cambridge, 1982.

[DY69] E. B. Dynkin, A. A. Yushkevich, *Markov processes: Theorems and problems*, Plenum, New York, 1969.

[Fe56] W. Feller, *Boundaries induced by non-negative matrices*, Trans. Amer. Math. Soc. **83** (1956), 19–54.

[FM97] B. Farb, L. Mosher, *A rigidity theorem for the solvable Baumslag–Solitar groups*, Invent. Math. (1997).

[Fo75] S. R. Foguel, *Iterates of a convolution on a non-abelain group*, Ann. Inst. H. Poincaré Séct. B **11** (1975), 199–202.

[Fu63a] H. Furstenberg, *A Poisson formula for semi-simple Lie groups*, Ann. of Math. **77** (1963), 335–386.

[Fu63b] H. Furstenberg, *Non-commuting random products*, Trans. Amer. Math. Soc. **108** (1963), 377-428.

[Fu71] H. Furstenberg, *Random walks and discrete subgroups of Lie groups*, Advances in Probability and Related Topics, vol. 1, Dekker, New York, 1971, pp. 3–63.

[Fu73] H. Furstenberg, *Boundary theory and stochastic processes on homogeneous spaces*, Proc. Symp. Pure Math., vol. 26, AMS, Providence R. I., 1973, pp. 193–229.

[GH90] E. Ghys, P. de la Harpe (eds.), *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Birkhäuser, Basel, 1990.

[Gi85] R. I. Grigorchuk, *The growth degrees of finitely generated groups and the theory of invariant means*, Math. USSR Izv. **48** (1985), 259–300.

[GM87] F. M. Gehring, G. J. Martin, *Discrete quasiconformal groups. I*, Proc. London Math. Soc. **55** (1987), 331–358.

[GM89] I. Ya. Goldsheid, G. A. Margulis, *Lyapunov exponents of a product of random matrices*, Russian Math. Surveys **44**:5 (1989), 11–71.

[Go76] R. W. Goodman, *Nilpotent Lie Groups*, Lecture Notes in Math., vol. 562, Springer, Berlin, 1976.

[Gr81] M. Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. IHES **53** (1981), 53–73.

[Gr87] M. Gromov, *Hyperbolic groups*, Essays in Group theory (S. M. Gersten, ed.), MSRI Publ., vol. 8, Springer, New York, 1987, pp. 75–263.

[GR85] Y. Guivarc'h, A. Raugi, *Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence*, Z. Wahrscheinlichkeitsth. Verw. Geb. **69** (1985), 187–242.

[GR89] Y. Guivarc'h, A. Raugi, *Propriétés de contraction d'un semi-groupe de matrices inversibles. Coefficients de Liapunoff d'un produit de matrices aléatoires indépendantes*, Israel J. Math. **65** (1989), 165–196.

[Gu73] Y. Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France **101** (1973), 333–379.

[Gu80a] Y. Guivarc'h, *Sur la loi des grands nombres et le rayon spectral d'une marche alèatoire*, Astérisque **74** (1980), 47–98.

[Gu80b] Y. Guivarc'h, *Quelques propriétés asymptotiques des produits de matrices aléatoires*, Springer Lecture Notes in Math. **774** (1980), 177–250.

[Ja95] W. Jaworski, *On the asymptotic and invariant σ-algebras of random walks on locally compact groups*, Probab. Theory Relat. Fields **101** (1995), 147–171.

[JP96] N. James, Y. Peres, *Cutpoints and exchangeable events for random walks*, Theory Probab. Appl. **41** (1996).

[Ka83] V. A. Kaimanovich, *The differential entropy of the boundary of a random walk on a group*, Russian Math. Surveys **38**:5 (1983), 142–143.

[Ka85a] V. A. Kaimanovich, *Examples of non-commutative groups with non-trivial exit boundary*, J. Soviet Math. **28** (1985), 579–591.

[Ka85b] V. A. Kaimanovich, *An entropy criterion for maximality of the boundary of random walks on discrete groups*, Soviet Math. Dokl. **31** (1985), 193–197.

[Ka89] V. A. Kaimanovich, *Lyapunov exponents, symmetric spaces and multiplicative ergodic theorem for semi-simple Lie groups*, J. Soviet Math. **47** (1989), 2387–2398.

[Ka91] V. A. Kaimanovich, *Poisson boundaries of random walks on discrete solvable groups*, Proceedings of Conference on Probability Measures on Groups X (Oberwolfach, 1990) (H. Heyer, ed.), Plenum, New York, 1991, pp. 205–238.

[Ka92] V. A. Kaimanovich, *Measure-theoretic boundaries of Markov chains, 0–2 laws and entropy*, Proceedings of the Conference on Harmonic Analysis and Discrete Potential Theory (Frascati, 1991) (M. A. Picardello, ed.), Plenum, New York, 1992, pp. 145–180.

[Ka94] V. A. Kaimanovich, *The Poisson boundary of hyperbolic groups*, C. R. Ac. Sci. Paris, Sér. I **318** (1994), 59–64.

[Ka95] V. A. Kaimanovich, *The Poisson boundary of covering Markov operators*, Israel J. Math **89** (1995), 77–134.

[Ka96] V. A. Kaimanovich, *Boundaries of invariant Markov operators: the identification problem*, Ergodic Theory of $\mathbb{Z}^d$-Actions (Proceedings of the Warwick Symposium 1993-4, M. Pollicott, K. Schmidt, eds.), London Math. Soc. Lecture Note Series, vol. 228, Cambridge Univ. Press, 1996, pp. 127–176.

[Ka97] V. A. Kaimanovich, *Hausdorff dimension of the harmonic measure on trees*, Ergod. Th. & Dynam. Sys. (1997).

[KM79] M. I. Kargapolov, Ju. I. Merzljakov, *Fundamentals of the theory of groups*, Graduate Texts in Math., vol. 62, Springer-Verlag, New York, 1979.

[KM96] V. A. Kaimanovich, H. Masur, *The Poisson boundary of the mapping class group*, Invent. Math. **125** (1996), 221–264.

[KP72] Yu. I. Kifer, S. A. Pirogov, *The decomposition of quasi-invariant measures into ergodic components*, Uspehi Mat. Nauk **27**:5 (1972), 239–240. (Russian)

[Kr85] U. Krengel, *Ergodic Theorems*, de Gruyter, Berlin, 1985.

[KV83] V. A. Kaimanovich, A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Prob. **11** (1983), 457–490.

[Le83] F. Ledrappier, *Une relation entre entropie, dimension et exposant pour certaines marches aléatoires*, C. R. Acad. Sci. Paris, Sér. I **296** (1983), 369–372.

[Le85] F. Ledrappier, *Poisson boundaries of discrete groups of matrices*, Israel J. Math. **50** (1985), 319–336.

[Le97] F. Ledrappier, *Some asymptotic properties of random walks on free groups*, CRM Proceedings and Lecture Notes (to appear).

[LM71] B. Ya. Levit, S. A. Molchanov, *Invariant Markov chains on a free group with a finite number of generators*, in Russian (translated in English in Moscow Univ. Math. Bull.), Vestnik Moscow Univ. **26**:4 (1971), 80–88.

[LPP96] R. Lyons, R. Pemantle, Y. Peres, *Random walks on the lamplighter group*, Ann. Probab. **24** (1996), 1993–2006.

[Ma91] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer-Verlag, Berlin, 1991.

[Mi68] J. Milnor, *A note on curvature and fundamental group*, J. Diff. Geom. **2** (1968), 1–7.

[Mo64] C. C. Moore, *Compactifications of symmetric spaces. I*, Amer. J. Math. **86** (1964), 201–218.

[Mo73] G. D. Mostow, *Strong Rigidity of Locally Symmetric Spaces*, Ann. of Math. Studies, vol. 78, Princeton Univ. Press, Princeton New Jersey, 1973.

[MP91] T. S. Mountford, S. C. Port, *Representations of bounded harmonic functions*, Ark. Mat. **29** (1991), 107–126.

[Pa87] S. J. Patterson, *Lectures on measures on limit sets of Kleinian groups*, Analytical and Geometric Aspects of Hyperbolic Space (D. B. A. Epstein, ed.), London Math. Soc. Lecture Note Series, vol. 111, Cambridge Univ. Press, 1987, pp. 281–323.

[PW87] M. A. Picardello, W. Woess, *Martin boundaries of random walks: ends of trees and groups*, Trans. Amer. Math. Soc. **302** (1987), 185–205.

[Ra77] A. Raugi, *Fonctions harmoniques sur les groupes localement compacts à base dénombrable*, Bull. Soc. Math. France. Mémoire **54** (1977), 5–118.

[Re84] D. Revuz, *Markov Chains*, 2nd revised ed., North-Holland, Amsterdam, 1984.

[Ro67] V. A. Rokhlin, *Lectures on the entropy theory of measure preserving transformations*, Russian Math. Surveys **22**:5 (1967), 1–52.

[Ro81] J. Rosenblatt, *Ergodic and mixing random walks on locally compact groups*, Math. Ann. **257** (1981), 31–42.

[Ru80] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer, Berlin, 1980.

[Sc78] K. Schmidt, *A probabilistic proof of ergodic decomposition*, Sankhyã, Ser. A **40** (1978), 10–18.

[Se83] C. Series, *Martin boundaries of random walks on Fuchsian groups*, Israel J. Math. **44** (1983), 221–242.

[Sg83] D. Segal, *Polycyclic groups*, Cambridge Univ. Press, Cambridge, 1983.

[St71] J. Stallings, *Group Theory and Three-Dimensional Manifolds*, Yale Univ. Press, New Haven, CT, 1971.

[Su79] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publ. Math. IHES **50** (1979), 171–202.

[Va85] N. Th. Varopoulos, *Long range estimates for Markov chains*, Bull. Sc. Math. **109** (1985), 225–252.

[Va86] N. Th. Varopoulos, *Information theory and harmonic functions*, Bull. Sci. Math. **110** (1986), 347–389.

[Wi90] G. Willis, *Probability measures on groups and some related ideals in group algebras*, J. Funct. Anal. **92** (1990), 202–263.

[Wo89] W. Woess, *Boundaries of random walks on graphs and groups with infinitely many ends*, Israel J. Math. **68** (1989), 271–301.

[Wo93] W. Woess, *Fixed sets and free subgroups of groups acting on metric spaces*, Math. Z. **214** (1993), 425–440.

[Zi78] R. J. Zimmer, *Amenable ergodic group actions and an application to Poisson boundaries of random walks*, J. Funct. Anal. **27** (1978), 350–372.