Scaling limits and aging for asymmetric trap models on the complete graph and K-processes

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Abstract

We obtain scaling limit results for asymmetric trap models and their infinite volume counterparts, namely asymmetric K processes. Aging results for the latter processes are derived therefrom.

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1 Introduction

The long time behavior of trap models and related processes with disordered parameters has been the theme of several papers in the recent literature. From the inaugurating work of Bouchaud [1], where the case of the complete graph was shown to exhibit *aging*, the same as well as other cases were analysed. The model on the complete graph was further studied in [2] and [3], with different points of view, and considering distinct time scales. And more recently, [4] took up the asymmetric case, which is also the model we study here.

The trap model in the complete graph is sometimes also called REM-like trap model, due to its resemblance to a dynamics for the Random Energy Model (REM [5]). Such a dynamics for the REM, on the hypercube rather than the complete graph, was studied in [6, 7], where aging results comparable to the ones of Bouchaud were derived. See also [8, 9, 10]. Trap and

trap-like models associated to correlated energy (mean field) spin glasses have been the object of more recent work: a dynamics for the p-spin model was studied in [11, 12], and results on the GREM-like trap model were obtained in [13].

Trap models on $\mathbb{Z}^d$ have also attracted a lot of interest, in connection with aging as well as with *localization*; see [14, 15, 16, 17, 18] – results on the asymmetric case were obtained recently in [22, 20, 21]. Analyses on tori were performed in [22, 23].

In this paper, we revisit the trap model in the complete graph, described briefly below in this introduction, and in full in Section 3. Our goal is twofold:

1. to propose a representation of the model – in terms of trap depth, rather than location – for which scaling limits can be derived in a unified manner in different scaling regimes;

2. and to introduce the infinite volume processes which result from these scaling limits, in particular the asymmetric K process.

Let us now briefly describe the asymmetric trap model in the complete graph with n vertices. This is a continuous time Markov chain on the vertices of that graph, whose mean jump time at site x is given by τ_x^{1-a}, where $a \in [0,1]$ is an asymmetry parameter, and whose transition probability from any site x to any site y is proportional to τ_y^a, where $\{\tau_x\}$ are iid positive random variables in the domain of attraction of an α-stable law. The random variable τ_x^{1-a} may be interpreted as the depth of the trap at site x. The case $a = 0$ is that of the *symmetric* model. We call the general case where $a \in [0,1]$ the *asymmetric* model. Let $Y_n(t)$ denote the site visited at time t.

This paper is more immediately related to [3] and [4], so let us briefly outline our results here against the background of the ones of those papers. Whereas in the former reference a scaling limit was derived for the symmetric model at times of the order of the deepest trap in the landscape, and then aging results were derived for a class of two-time correlation functions of the limit model at vanishing times, in here we present similar limit results for the asymmetric model. Rather than looking at $Y_n(t)$ however, we consider $Z_n(t) = \tau_{Y_n(t)}^{1-a}$, the depth of the currently visited trap. As explained below, this is a convenient representation for taking scaling limits, not only at times of the order of the deepest trap in the landscape, which we do here using this representation (see Theorem 3.1), obtaining a limiting process which we denote by Z, but at shorter time scales as well. We call Z the *asymmetric K process*, in allusion to the K process introduced in [3]. We further derive a scaling limit result for Z at vanishing times (see Theorem 3.2), obtaining a limiting process $\hat{Z}$ which is self similar of index 1. The latter fact may be interpreted as a fuller aging result for Z, involving the dynamics itself, not only a class of correlation functions thereof. Other scaling regimes of Z_n may be analysed with the same approach, with similar results.

Scaling limits of asymmetric trap models in the complete graph are also the main theme of [4]. In that work scaling limits of the *clock process* are derived in several scaling regimes (essentially all of them: from "order 1", where the volume limit is taken first, and then the time limit, to the scale where the model is virtually at equilibrium, including scales in between, in particular the ones treated here); occurrence of aging and other dynamical phenomena are discussed for each regime.

One reason to consider a representation like Z_n, as we do here, rather than the clock process, is that, besides the information on the jump times given by the latter process, Z_n provides also

location information, absent in that process. For, say, correlation functions which depend only on jump times (like the Π functions discussed on Subsection 3.3 below; see (3.59-3.62)), the clock process is enough. But other ones require location information, and in those cases the clock process is no longer enough on its own. We discuss two such examples in Subsection 3.3 below.

Z_n and Z, as well as their rescaled versions, and $\hat{Z}$ also, can be described as functions of two related subordinators, the second being obtained as the integral of an independent iid family of mean 1 exponentials with respect to the first one. Once we obtain the limit of the first subordinator in a given scaling regime, a continuity property of the above mentioned function implies a limit result for the original process. Section 2 below is devoted to establishing that continuity property (see Lemma 2.1) in a somewhat abstract setting, which may turn out to be the setting of similar processes of interest.

In Section 3 we describe our trap models and K processes in more detail and then, applying the auxiliary result of Section 2, we derive scaling limit results for them, as anticipated above, the one for the trap model in Subsection 3.1, and the one for the K process in Subsection 3.2. In the closing Subsection 3.3 we discuss the derivation of aging results for three particular two-time correlation functions of Z as a corollary to Theorem 3.2.

2 A continuity lemma about a class of trajectories in D

Let D be the space of càdlàg real trajectories on $\mathbb{R}^+ = [0, \infty)$ equipped with the J_1 Skorohod metric (see e.g. [24] Chapter 3, Section 5). Let $\mathbb{N}^* = \{1, 2, \ldots\}$ denote the positive integers.

Let S, S^ε, $\varepsilon > 0$, be nonnegative nondecreasing jump functions in D, i.e., suppose that there exist (countable) subsets $A^\varepsilon = \{x_i^\varepsilon, i \in \mathbb{N}^*\}$ and $A = \{x_i, i \in \mathbb{N}^*\}$ of $\mathbb{R}^+$ and positive number sequences $\{\gamma_{x_i^\varepsilon}^\varepsilon, i \in \mathbb{N}^*\}$ and $\{\gamma_{x_i}, i \in \mathbb{N}^*\}$ such that

$$S_r^\varepsilon = \sum_{i:\, x_i^\varepsilon \in [0,r]} \gamma_{x_i^\varepsilon}^\varepsilon < \infty, \quad S_r = \sum_{i:\, x_i \in [0,r]} \gamma_{x_i} < \infty, \quad r \geq 0. \tag{2.1}$$

Consider $\{T_i, i \in \mathbb{N}^*\}$, a family of i.i.d. exponential random variables of mean 1 and let

$$\Gamma_r^\varepsilon = \sum_{i:\, x_i^\varepsilon \in [0,r]} \gamma_{x_i^\varepsilon}^\varepsilon T_i, \quad \Gamma_r = \sum_{i:\, x_i \in [0,r]} \gamma_{x_i} T_i, \quad r \geq 0, \tag{2.2}$$

$$Z_t^\varepsilon = \begin{cases} \gamma_{x_{i_0}^\varepsilon}^\varepsilon, & \text{if } t \in [\Gamma_{x_{i_0}^\varepsilon -}^\varepsilon, \Gamma_{x_{i_0}^\varepsilon}^\varepsilon) \text{ for some } i_0, \\ 0, & \text{if } t \notin [\Gamma_{x_i^\varepsilon -}^\varepsilon, \Gamma_{x_i^\varepsilon}^\varepsilon) \text{ for any } i, \end{cases} \tag{2.3}$$

and

$$Z_t = \begin{cases} \gamma_{x_{i_0}}, & \text{if } t \in [\Gamma_{x_{i_0} -}, \Gamma_{x_{i_0}}) \text{ for some } i_0, \\ 0, & \text{if } t \notin [\Gamma_{x_i -}, \Gamma_{x_i}) \text{ for any } i. \end{cases} \tag{2.4}$$

Lemma 2.1 *Let S^ε, S, Z^ε, Z be as above. If $S^\varepsilon \to S$, then $Z^\varepsilon \to Z$ in distribution.*

Remark 1 *The probability measure in the statement of the above lemma is the one underlying $\{T_i, i \in \mathbb{N}^*\}$.*

Remark 2 *From (2.1-2.4), we see that $Z = \Xi(S, \{T_i,\ i \in \mathbb{N}^*\})$ and $Z^\varepsilon = \Xi(S^\varepsilon, \{T_i,\ i \in \mathbb{N}^*\})$, where Ξ is the composition underlying the above definitions. Lemma 2.1 then establishes a continuity property of Ξ in its first argument.*

Proof of Lemma 2.1

We will assume that there exists $R' \in \mathbb{R}^+$ such that $|A \cap [0, R']| = \infty$. Other cases may be argued similarly, with simpler arguments.

Let Γ^{-1} be the (right continuous) inverse of Γ. Let us fix $T > 0$. Then one readily checks that, given $\delta > 0$, there exists $R \notin A$, $R \geq R'$, such that

$$\mathbb{P}(\Gamma^{-1}(T) \geq R) \leq \delta. \tag{2.5}$$

Given $\eta > 0$, we may choose $\delta' > 0$ be such that

$$S_{R+\delta'} - S_R < \eta. \tag{2.6}$$

Let us now enumerate $A \cap [0, R] = \{x_1, x_2, \ldots\}$ such that $\gamma_{x_1} \geq \gamma_{x_2} \geq \ldots$. From the hypothesis, there exists $m = m(\varepsilon)$, with $m \to \infty$ as $\varepsilon \to 0$, and an enumeration of $A^\varepsilon \cap [0, R] = \{x_1^\varepsilon, x_2^\varepsilon, \ldots\}$ such that as $\varepsilon \to 0$

$$\left(\sup_{1 \leq i \leq m} |x_i^\varepsilon - x_i| \right) \vee \left(m \sup_{1 \leq i \leq m} |\gamma_{x_i^\varepsilon}^\varepsilon - \gamma_{x_i}| \right) \to 0. \tag{2.7}$$

It follows from this and the hypothesis that, given $\eta > 0$, for all small enough ε and $1 \leq k \leq m$

$$\sum_{i>k} \gamma_{x_i^\varepsilon}^\varepsilon = S_R^\varepsilon - \sum_{i=1}^{k} \gamma_{x_i^\varepsilon}^\varepsilon \leq S_{R+\delta'} - \sum_{i=1}^{k} \gamma_{x_i} + \eta = S_{R+\delta'} - S_R + \sum_{i>k} \gamma_{x_i} + \eta \leq \sum_{i>k} \gamma_{x_i} + 2\eta. \tag{2.8}$$

We now recall that in the J_1 topology, functions are close if they are uniformly close inside arbitrary bounded intervals, after allowing small time distortions (for details see e.g. [24] Chapter 3, Section 5).

Now, given $k \geq 1$ arbitrary but fixed, independent of ε, let $\{\bar{x}_1, \ldots, \bar{x}_k\}$ be an enumeration of $\{x_1, \ldots, x_k\}$ such that $\{\bar{x}_1 < \ldots < \bar{x}_k\}$. This leads to an enumeration $\{\bar{x}_1^\varepsilon, \ldots, \bar{x}_k^\varepsilon\}$ of $\{x_1^\varepsilon, \ldots, x_k^\varepsilon\}$ such that for $1 \leq i \leq k$

$$\bar{x}_i^\varepsilon \to \bar{x}_i \text{ and } \gamma_{\bar{x}_i^\varepsilon}^\varepsilon \to \gamma_{\bar{x}_i} \tag{2.9}$$

(see paragraph of (2.7) above). At this point we relabel $\{T_i\}$ so that $T_1, \ldots, T_k$ are attached to $\bar{x}_1 < \ldots < \bar{x}_k$ and commonly to $x_1^\varepsilon, \ldots, x_k^\varepsilon$, respectively, which does not change distributions. Let $Z^{(k)}$ and $Z^{(k,\varepsilon)}$ be the respective versions of Z and Z^ε with the relabeled $\{T_i\}$.

Let us now take a family of temporal distortions $(\lambda^\varepsilon) = (\lambda_k^\varepsilon)$ as follows. For $1 \leq i \leq k$, we consider the time intervals $I_i = [t_i^-, t_i]$, where $t_i = \Gamma_{\bar{x}_i}$ and $t_i^- = \Gamma_{\bar{x}_i-}$, and $[t_i^{\varepsilon-}, t_i^\varepsilon]$, where $t_i^\varepsilon = \Gamma_{\bar{x}_i^\varepsilon}^\varepsilon$ and $t_i^{\varepsilon-} = \Gamma_{\bar{x}_i^\varepsilon-}^\varepsilon$, and then define

$$\lambda^\varepsilon(s) = \begin{cases} \frac{t_1^{\varepsilon-}}{t_1^-} s, & \text{if } 0 \leq s \leq t_1^-, \\ \frac{t_i^\varepsilon - t_i^{\varepsilon-}}{t_i - t_i^-}(s - t_i^-) + t_i^{\varepsilon-}, & \text{if } t_i^- \leq s \leq t_i, \\ \frac{t_{i+1}^{\varepsilon-} - t_i^\varepsilon}{t_{i+1}^- - t_i}(s - t_i) + t_i^\varepsilon, & \text{if } t_i \leq s \leq t_{i+1}^-, \\ (s - t_{k+1}^-) + t_{k+1}^{\varepsilon-}, & \text{if } s \geq t_{k+1}^-, \end{cases} \tag{2.10}$$

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where $t_{k+1}^- := \Gamma_R$, $t_{k+1}^{\varepsilon-} := \Gamma_R^\varepsilon$.

At this point, we have two tasks: the first one is to control the slopes of the functions λ^ε and the second one is to control the sup norm of the difference $Z_{\lambda^\varepsilon(t)}^{(k,\varepsilon)} - Z_t^{(k)}$.

We start by the second task. Let $\mathcal{M} = \cup_{i=1}^k I_i$. If $t \in \mathcal{M}$, then

$$|Z_t^{(k)} - Z_{\lambda^\varepsilon(t)}^{(k,\varepsilon)}| \leq \max_{1 \leq i \leq k} |\gamma_{\bar{x}_i} - \gamma_{\bar{x}_i^\varepsilon}^\varepsilon|, \tag{2.11}$$

which goes to zero as ε goes to zero by (2.9).

If $t \in [0, t_{k+1}^-] \setminus \mathcal{M}$, then we have that $Z_t^{(k)} \leq \gamma_{x_{k+1}}$ and $Z_{\lambda^\varepsilon(t)}^{(k,\varepsilon)} \leq \max_{i>k} \gamma_{x_i^\varepsilon}^\varepsilon$. Hence,

$$|Z_t^{(k)} - Z_{\lambda^\varepsilon(t)}^{(k,\varepsilon)}| \leq \gamma_{x_{k+1}} \vee \max_{i>k} \gamma_{x_i^\varepsilon}^\varepsilon \leq \gamma_{x_{k+1}} \vee \sum_{i>k} \gamma_{x_i^\varepsilon}^\varepsilon \leq \sum_{i>k} \gamma_{x_i} + 2\eta, \tag{2.12}$$

for all small enough ε, by (2.8).

Now, we solve the first problem by considering two cases:

1) If $s \in [t_i^-, t_i]$ for some $1 \leq i \leq k$, then the slope of λ^ε is given by

$$\frac{t_i^\varepsilon - t_i^{\varepsilon-}}{t_i - t_i^-} = \frac{\gamma_{\bar{x}_i^\varepsilon}^\varepsilon T_i}{\gamma_{\bar{x}_i} T_i} = \frac{\gamma_{\bar{x}_i^\varepsilon}^\varepsilon}{\gamma_{\bar{x}_i}} \to 1 \tag{2.13}$$

as $\varepsilon \to 0$, by (2.9).

2) If $s \in [t_i, t_{i+1}^-]$ for some $0 \leq i \leq k$, where $t_0 := 0$, then it suffices to prove that

$$t_i^\varepsilon \to t_i \tag{2.14}$$
$$t_i^{\varepsilon-} \to t_i^- \tag{2.15}$$

as $\varepsilon \to 0$ in probability.

In all cases, the absolute value of the difference of right and left hand sides is bounded above by

$$\sum_{i=1}^m |\gamma_{x_i^\varepsilon}^\varepsilon - \gamma_{x_i}| T_i + \sum_{i>m} |\gamma_{x_i^\varepsilon}^\varepsilon - \gamma_{x_i}| T_i. \tag{2.16}$$

The first term vanishes almost surely as $\varepsilon \to 0$ by (2.7) and the Law of Large Numbers, and, given $\eta > 0$, the expected value of the second term is bounded above by

$$\sum_{i>m} \gamma_{x_i^\varepsilon}^\varepsilon + \sum_{i>m} \gamma_{x_i} \leq 2 \sum_{i>m} \gamma_{x_i} + \eta, \tag{2.17}$$

for all small enough ε, where use is made of (2.8) in the latter inequality, and (2.14,2.15) follow since $m \to \infty$ as $\varepsilon \to 0$ and η is arbitrary.

To conclude, given $0 < \zeta < 1, \delta > 0$, choose $T > -\log(\zeta/2)$, and then R satisfying (2.5), and then δ' satisfying (2.6) with $\eta = \zeta/4$, and then k such that $\sum_{i>k} \gamma_{x_i} < \zeta/2$. Choosing now λ^ε as in (2.10), we conclude that

$$\limsup_{\varepsilon \to 0} \mathbb{P}(d(Z^{(k,\varepsilon)}, Z^{(k)}) > \zeta) \leq \delta, \tag{2.18}$$

where d is the J_1 Skorohod distance on D (see [24] Chapter 3, Section 5). Since $Z^{(k,\varepsilon)} = Z^\varepsilon$ and $Z^{(k)} = Z$ in distribution for all fixed k and ε small enough, the result follows. $\square$

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Let us now explain how Lemma 2.1 will be used in the sequel. Our aim is to apply it to a case where S^ε and S are random objects, in fact subordinators, with parameters that are themselves random, which we call *environment*. Both S^ε and S, as well as their respective environments, will be independent of $\{T_i\}$, and the convergence $S^\varepsilon \to S$ will hold only in distribution: either 1) the joint distribution of the environment and the subordinators, or 2) the distributions of subordinators given the environment, for almost every realization of the environment. In both cases, we may use the Skorohod representation theorem (see e.g. [25] Theorem 3.2.2). In case 1) we will first explicitly choose a convenient version of the environment, for which the distribution of the subordinator, given the environment, converges for almost every realization of the environment; with the modified environment, we are effectively in case 2. We can then, by Skorohod representation, in both cases, for each choice of the environment, choose versions of the subordinators that converge almost surely, and then we are in the setting of Lemma 2.1. It is clear that the conclusion of the lemma holds for the original subordinator, where the distribution referred to in the lemma is the joint distribution of $\{T_i\}$ and the subordinators given the original environment in case 2, and the modified environment in case 1, for almost every realization of that environment in each case. In case 1, the result of the lemma will then hold for the overall joint distribution of $\{T_i\}$, the subordinators given the environment, and the environment.

Establishing the convergence in distribution of the subordinators is done by verifying the convergence of the respective Laplace exponents.

3   Application to trap models on the complete graph and K processes

We will apply the lemma above to show scaling limit results for trap models in the complete graph and for K-processes. We introduce these two processes next.

We first consider the trap model on the complete graph

$$K_n = \{\{1, \ldots, n\}, \{(x, y), x, y = 1, \ldots, n\}\} \tag{3.1}$$

with n vertices (differently from the usual definition, here we include self loops, for convenience – this should not matter in the convergence results below): $Y_n = (Y_n(t))_{t \geq 0}$, which is a continuous time Markov chain with jump rate at site x given by

$$\tau_x^{-(1-a)}, \tag{3.2}$$

and transition probability from site x to site y given by

$$\frac{\tau_y^a}{\sum_{z=1}^n \tau_z^a}, \tag{3.3}$$

where $a \in [0, 1]$ is a parameter, and $\tau := \{\tau_x, x \in K_n\}$ is an independent family of positive random variables with common distribution in the domain of attraction of a stable law of degree $0 < \alpha < 1$, that is,

$$\mathbb{P}(\tau_1 > t) = \frac{L(t)}{t^\alpha}, \ t > 0, \tag{3.4}$$

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where L is a slowly varying function at infinity.

We call Y_n an *asymmetric* or *weighted trap model on the complete graph* with asymmetry parameter a, mean jump time parameters $\{\tau_x^{1-a}, x = 1, \ldots, n\}$ and weights $\{\tau_x^a, x = 1, \ldots, n\}$. The latter set of parameters may indeed be seen as unnormalized weights of the transition probabilities of Y_n. Notice that the $a = 0$ (symmetric) case corresponds to uniform weights.

We will consider the following construction of Y_n. Let

$$\mathcal{N} = \{N^{(x)} := (N_r^{(x)})_{r \geq 0}, \, x \in \mathbb{N}^*\} \tag{3.5}$$

be a family of independent Poisson counting processes such that the rate of $N^{(x)}$ is τ_x^a. Let $\sigma_j^{(x)}$ the j-th event time of $N^{(x)}$, $j \geq 1$. Let also

$$\mathcal{T} = \{T_0; \, T_i^{(x)}, \, x \in \mathbb{N}^*\} \tag{3.6}$$

be independent mean 1 exponential random variables, independent of $\mathcal{N}$ and τ, and define for $r \geq 0$

$$S_n(r) = \sum_{x=1}^{n} \tau_x^{1-a} \, N_r^{(x)}, \quad \Gamma_n(r) = \sum_{x=1}^{n} \tau_x^{1-a} \sum_{i=1}^{N_r^{(x)}} T_i^{(x)}. \tag{3.7}$$

Then

$$Y_n(t) = x, \text{ if } \Gamma_n(\sigma_j^{(x)}-) \leq t < \Gamma_n(\sigma_j^{(x)}) \text{ for some } x, j \geq 1. \tag{3.8}$$

is a construction of Y_n as above described, with initial state distributed on $\{1, \ldots, n\}$ in such a way that site x has probability weight proportional to τ_x^a, $x \in \{1, \ldots, n\}$.

Remark 3 *Regarding the latter point, notice that the initial state of Y_n is the one whose Poisson mark is the earliest, so it corresponds to the minimum of n independent exponential random interarrival times with rates τ_x^a, $x \in \{1, \ldots, n\}$, and it is well known that the probability that the minimum of n independent exponential random variables is a given such random variable is proportional to its rate.*

We are interested below in

$$Z_n(t) = \tau_{Y_n(t)}^{1-a}. \tag{3.9}$$

This is the representation for the process aluded to at the introduction above. It has been considered in [2], where the symmetric ($a = 0$) case was studied, and a (single time) scaling limit result was derived for it, first taking the volume, and the time, to infinity (see Proposition 2.10 in that reference) – this is an aging regime not considered in this paper, but rather in [4].

Remark 4 *Z_n and Y_n may be seen as processes in random environment, where τ is the set of random parameters acting as environment. Indeed, given τ, both are Markovian (this should be clear for Y_n, but a moment's thought reveals that it is true for Z_n as well, even when there are same values for τ_i's with distinct i's). Notice also that τ is an environment for S_n as well, which for each $n \geq 1$ is a subordinator for every fixed such environment (recall the discussion at the end of Section 2.1). This aspect, which is characteristic of the complete graph, makes our approach particularly suitable, since by an application of (the continuity) Lemma 2.1, we are left with establishing convergence of subordinators (in the Skorohod topology), which reduces to showing convergence of Laplace exponents (in the topology of real numbers), which is relatively simple, as we will see below.*

Remark 5 *Given S_n, Z_n may be identified in distribution to $\Xi(S_n, \{T_i, i \in \mathbb{N}^*\})$, with Ξ introduced in Remark 2.*

We now turn to K-processes, which is a Markov process in continuous time on $\bar{\mathbb{N}}^* = \{1, 2, \ldots, \infty\}$ constructed in a similar way as Y_n was above, as follows. Let $\gamma = \{\gamma_x, x \in [0, \infty)\}$ be the increments of an α-stable subordinator in $[0, \infty)$ given by a Poisson process $\mathcal{P}$ in $(0, \infty) \times (0, \infty)$ with intensity measure

$$\alpha x^{-1-\alpha} \, dx \, dy. \tag{3.10}$$

Let

$$\hat{\mathcal{N}} = \{\hat{N}^{(x)} := (\hat{N}_r^{(x)})_{r \geq 0}, \, x \in \mathbb{N}^*\} \tag{3.11}$$

be a family of independent Poisson counting processes such that the rate of $\hat{N}^{(x)}$ is γ_x^a. Let $\hat{\sigma}_j^{(x)}$ the j-th event time of $\hat{N}^{(x)}$, $j \geq 1$. Let $\mathcal{T}$ be as above (see (3.6)), and independent of $\hat{\mathcal{N}}$.

Define for $r \geq 0$

$$S(r) = \sum_{x \in [0,1]} \gamma_x^{1-a} \hat{N}_r^{(x)}, \quad \Gamma(r) = \sum_{x \in [0,1]} \gamma_x^{1-a} \sum_{i=1}^{\hat{N}_r^{(x)}} T_i^{(x)}, \tag{3.12}$$

and then make

$$Y_t = \begin{cases} x, & \text{if } \Gamma(\hat{\sigma}_j^{(x)}-) \leq t < \Gamma(\hat{\sigma}_j^{(x)}) \text{ for some } x, j \geq 1, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.13}$$

Remark 6 *It can be verified that when $a > \alpha$, then Y is a jump process, and so there is almost surely no t for which $Y(t) = \infty$ (since in this case $\cup_{j,x}[\Gamma(\hat{\sigma}_j^{(x)}-), \Gamma(\hat{\sigma}_j^{(x)})) = [0, \infty)$). And in the case where $a \leq \alpha$, there almost surely exist t's for which $Y(t) = \infty$. (One way to check these claims is by verifying that when $a > \alpha$, $\{\hat{\sigma}_j^{(x)}; j \geq 1, x \in [0, 1]\}$ is a discrete subset of $[0, \infty)$ almost surely, and when $a \leq \alpha$, it is almost surely dense in $[0, \infty)$, and these in turn follow from the fact that $\sum_{x \in [0,1]} \gamma_x^a$ is almost surely finite in the former case, and infinite in the latter one.)*

Let

$$Z_t = \gamma_{Y_t}^{1-a}, \tag{3.14}$$

where γ_∞ should be interpreted as 0.

Remark 7 *Z and Y may be seen as processes in random environment, where γ (more specifically, $\gamma|_{[0,1]} = \{\gamma_x, x \in [0, 1]\}$) is the environment. Indeed, given γ, both are Markovian. $\gamma|_{[0,1]}$ is also an environment for S, which is a subordinator for every fixed such environment (recall the discussion at the end of Section 2.1).*

Remark 8 *Given S, Z may be identified in distribution to $\Xi(S, \{T_i, i \in \mathbb{N}^*\})$, with Ξ introduced in Remark 2.*

Remark 9 *In [3] and other references the representations used for the trap model and K process are the ones given here by $Y_n(t)$ and $Y(t)$, $t \geq 0$, respectively (see (3.8) and (3.13) above). The alternative representation $Z_n(t)$ and $Z(t)$, $t \geq 0$, we adopt here (see (3.9) and (3.14) above) has the advantage of leading to a unifying approach for taking the scaling limits of those processes, as explained in the introduction and will be done in detail in Subsections 3.1 and 3.2 below.*

In the next subsection, we will consider a particular scaling regime for Z_n and establish a scaling limit result under which Z_n converges to the K process. Then, in the following subsection we will derive a scaling limit result satisfied by Z. All proofs will rely on Lemma 2.1 above to get the results from the convergence of the appropriate S^ε in each case (see statement of that lemma and its preliminaries above). In order to obtain the latter convergence, since we have subordinators in all cases, it will suffice to establish convergence of the associated Laplace exponents. The last subsection is devoted to a discussion on aging results (for two-time correlation functions) satisfied by Z as a consequence of Theorem 3.2 and other results.

3.1 Scaling limit for Z_n at large times

For $r \geq 0$, let

$$U(r) = \sum_{x \in [0,r]} \gamma_x. \tag{3.15}$$

Given a sequence $(c_n)_{n \geq 1}$, set

$$Z_t^{(n)} = c_n^{1-a} Z_n(t/c_n^{1-a}), \ t \geq 0. \tag{3.16}$$

Theorem 3.1 *There exists a deterministic sequence $(c_n)_{n \geq 1}$ such that*

$$(Z_t^{(n)})_{t \geq 0} \to (Z_t)_{t \geq 0}. \tag{3.17}$$

in distribution as $n \to \infty$.

Remark 10 *The distribution referred to above is the joint one of the dynamical and environmental random variables. The sequence (c_n) will be exhibited explicitly in the proof below (see 3.21).*

Proof
By Lemma 2.1, and recalling the discussion at the end of Section 2, it is enough to establish the limit

$$S^{(n)} \to S, \tag{3.18}$$

where

$$S_r^{(n)} := c_n^{1-a} S_n(c_n^a r) = \sum_{x=1}^{n} (c_n \tau_x)^{1-a} N_{c_n^a r}^{(x)}, \tag{3.19}$$

since, given $S^{(n)}$, $Z^{(n)}$ is identically distributed with $\Xi(S^{(n)}, \{T_i, i \in \mathbb{N}^*\})$. (See the discussion at the end of Section 2 above.)

Since $S^{(n)}$, $n \geq 1$, and S are subordinators (given τ), it is enough to show the convergence of the corresponding Laplace exponents (see Corollary 3.6 page 374 in [26]). A straightforward computation yields

$$\varphi_n(\lambda) := \sum_{x=1}^{n} (c_n \tau_x)^a (1 - e^{-\lambda(c_n \tau_x)^{1-a}}), \; \lambda \geq 0, \tag{3.20}$$

as the Laplace exponent of $S^{(n)}$. In order to establish its limit (a weak one), we will make a precise choice of c_n and switch to another version of τ, so that the limit will be strong. We follow [14], Section 3. Let

$$c_n = (\inf\{t \geq 0 : \mathbb{P}(\tau_1 > t) \leq n^{-1}\})^{-1}, \tag{3.21}$$

$$\tau_x^{(n)} := c_n^{-1} g_n(U(x) - U(x - 1/n)), \; x \in (0,1] \cap \tfrac{1}{n}\mathbb{Z} \tag{3.22}$$

$$g_n(y) = c_n G^{-1}(n^{1/\alpha} y), \; y \geq 0, \tag{3.23}$$

where G^{-1} is the inverse of the function G defined by the following condition.

$$\mathbb{P}(U(1) > G(x)) = \mathbb{P}(\tau_1 > x), \; x \geq 0 \tag{3.24}$$

Then

$$\tilde{\varphi}_n(\lambda) := \sum_{x \in (0,1] \cap \frac{1}{n}\mathbb{Z}} (\tau_x^{(n)})^a (1 - e^{-\lambda(\tau_x^{(n)})^{1-a}}) \tag{3.25}$$

is identically distributed with $\varphi_n(\lambda)$ (simultaneously for all λ). Now let

$$\mathfrak{T}_\delta = \{x \in [0,1] : \gamma(x) > \delta\} = \{x_1 < \ldots < x_K\}, \tag{3.26}$$

and

$$\mathfrak{T}_\delta^{(n)} = \left\{ x_1^{(n)} = \frac{1}{n}\lceil nx_1 \rceil < \ldots < x_K^{(n)} = \frac{1}{n}\lceil nx_K \rceil \right\}, \tag{3.27}$$

where the strict inequalities in (3.27) hold provided n is large enough (for each fixed δ).

Now Lemma 3.1 in [14] implies that for every $\delta > 0$

$$\sum_{x \in \mathfrak{T}_\delta^{(n)}} (\tau_x^{(n)})^a (1 - e^{-\lambda(\tau_x^{(n)})^{1-a}}) \to \sum_{x \in \mathfrak{T}_\delta} \gamma_x^a (1 - e^{-\lambda \gamma_x^{1-a}}) \tag{3.28}$$

almost surely as $n \to \infty$. One also readily checks that

$$\sum_{x \in (0,1] \cap \frac{1}{n}\mathbb{Z} \setminus \mathfrak{T}_\delta^{(n)}} (\tau_x^{(n)})^a (1 - e^{-\lambda(\tau_x^{(n)})^{1-a}}) \leq \lambda \sum_{x \in (0,1] \cap \frac{1}{n}\mathbb{Z} \setminus \mathfrak{T}_\delta^{(n)}} \tau_x^{(n)}. \tag{3.29}$$

Since, as argued in paragraphs of (3.25-3.28) in [14], we have that the $\lim_{\delta \to 0} \limsup_{n \to \infty}$ of the sum in the right hand side of (3.29) vanishes almost surely, we may conclude that

$$\tilde{\varphi}_n(\lambda) \to \sum_{x \in [0,1]} \gamma_x^a (1 - e^{-\lambda \gamma_x^{1-a}}), \; \lambda \geq 0, \tag{3.30}$$

almost surely. The right hand side is the Laplace exponent of S, so the proof is complete.

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3.2 Scaling limit of Z at small times

In this subsection, we assume $0 \le a < \alpha$. Let

$$Z_t^{(\varepsilon)} = \varepsilon^{-1} Z_{\varepsilon t}. \tag{3.31}$$

Before stating a convergence result for $Z^{(\varepsilon)}$, let us describe the limit process. Let $(\hat{S}_t)_{t \ge 0}$ be an $\hat{\alpha}$-stable subordinator, where

$$\hat{\alpha} = \frac{\alpha - a}{1 - a}, \tag{3.32}$$

and whose Laplace exponent is given by $\hat{\varphi}(\lambda) = \hat{c}\lambda^{\hat{\alpha}}$, where $\hat{c}$ is a constant to be determined below.

We may then write $\hat{S}$ as a partial sum of its increments as follows.

$$\hat{S}_r = \sum_{x \in [0,r]} \hat{\gamma}_x, \tag{3.33}$$

where $\{\hat{\gamma}_x, \ x \in \mathbb{N}^*\}$ are the increments of $\hat{S}$.

Let now

$$\hat{\Gamma}_r = \sum_{x \in [0,r]} \hat{\gamma}_x T_x, \tag{3.34}$$

where $\{T_x, \ x \in [0, \infty)\}$ is an iid family of mean 1 exponential random variables, independent of $\hat{S}$.

Remark 11 *One may readily check that $\hat{\Gamma}$ is also an $\hat{\alpha}$-stable subordinator (under the joint distribution of $\hat{S}$ and $\{T_x, \ x \in [0, \infty)\}$).*

Now define

$$\hat{Z}_t = \begin{cases} \hat{\gamma}_x, & \text{if } t \in [\hat{\Gamma}_{x-}, \hat{\Gamma}_x) \text{ for some } x \in [0, \infty) \\ 0, & \text{for all other } t \ge 0, \text{ if any.} \end{cases} \tag{3.35}$$

Remark 12 *$\hat{Z}$ may be seen as a process in random environment, where $\hat{S}$ is the environment. Indeed, given $\hat{S}$, $\hat{Z}$ is Markovian. And the distribution of $\hat{Z}$ (integrated over the environment) makes it a self similar process of index 1, that is, $(\hat{Z}_t)_{t \ge 0} = (c^{-1}\hat{Z}_{ct})_{t \ge 0}$ in distribution for every constant $c > 0$. This latter property explains the aging behavior of Z in its small time scaling regime, as established below.*

Remark 13 *Given $\hat{S}$, $\hat{Z}$ may be identified in distribution to $\Xi(\hat{S}, \{T_i, \ i \in \mathbb{N}^*\})$, with Ξ introduced in Remark 2.*

We are now ready to state this subsection's result.

Theorem 3.2 *If $0 \le a < \alpha$ then for almost every γ*

$$(Z_t^{(\varepsilon)})_{t \ge 0} \to (\hat{Z}_t)_{t \ge 0}. \tag{3.36}$$

in distribution as $\varepsilon \to 0$.

Remark 14 *The distributions referred to above are, to the left, the conditional distribution of $(\hat{Z}_t^{(\varepsilon)})$ given γ, and to the right, the (unconditional) distribution of $(\hat{Z}_t)$.*

Remark 15 *An argument for the integrated version of Theorem 3.2 (where the stated weak convergence is with respect to the joint distribution of Z given γ and γ) is simpler than the one given next. See Remark 16 below.*

Proof

Let

$$\hat{S}_r^{(\varepsilon)} = \varepsilon^{-1} \sum_{x \in [0,1]} \gamma_x^{1-a} \hat{N}_{\varepsilon \hat{\alpha} r}^x, \ r \geq 0 \tag{3.37}$$

where $\hat{\alpha}$ was introduced in (3.32) above. Then, given γ and $\varepsilon > 0$, $(\hat{S}_t^{(\varepsilon)}, t \geq 0)$ is a subordinator, and its Laplace exponent equals

$$\hat{\varphi}^{(\varepsilon)}(\lambda) = \varepsilon^{\hat{\alpha}} \sum_{x \in [0,1]} \gamma_x^a (1 - e^{-\lambda \varepsilon^{-1} \gamma_x^{1-a}}), \ \lambda \geq 0. \tag{3.38}$$

By Lemma 2.1, and recalling the discussion at the end of Section 2, to get the result, it is enough to establish the limit

$$\hat{S}^{(\varepsilon)} \to \hat{S} \tag{3.39}$$

in distribution as $\varepsilon \to 0$ for a.e. γ. Since we are dealing with subordinators, it suffices to show that for almost every γ

$$\hat{\varphi}^{(\varepsilon)}(\lambda) \to \hat{c} \lambda^{\hat{\alpha}}, \ \lambda \geq 0, \tag{3.40}$$

as $\varepsilon \to 0$, for some positive constant finite $\hat{c}$. This is obvious for $\lambda = 0$, so let us fix $\lambda > 0$, and write

$$\lambda^{-\hat{\alpha}} \hat{\varphi}^{(\varepsilon)}(\lambda) = R^{-\alpha} \sum_{x \in [0,1]} (R \gamma_x)^a (1 - e^{-(R \gamma_x)^{1-a}}) \tag{3.41}$$

with $R = (\varepsilon^{-1} \lambda)^{\frac{1}{1-a}}$, and then argue in the sequel that the left hand side converges to a constant as $R \to \infty$ for a.e. γ.

We start by considering

$$W := R^{-\alpha} \sum_{x \in [0,1]} \sum_{i=1}^{R\delta^{-1}} (R \gamma_x)^a (1 - e^{-(R \gamma_x)^{1-a}}) \mathbb{I}_{\{\gamma_x \in [\frac{\delta}{R}(i-1), \frac{\delta}{R}i]\}}. \tag{3.42}$$

Since the difference between W and the left hand side of (3.41) is bounded above by

$$R^{-(\alpha-a)} \sum_{x \in [0,1]} \gamma_x^a \mathbb{I}_{\{\gamma_x > 1\}}, \tag{3.43}$$

which vanishes as $R \to \infty$ for a.e. γ, it is enough to establish the convergence result for W. We estimate it as follows.

$$W - X_1 \ \leq \ R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} X_i^+ := R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} (\delta i)^a (1 - e^{-(\delta i)^{1-a}}) M_i \tag{3.44}$$

$$W \ \geq \ R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} X_i^- := R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} (\delta(i-1))^a (1 - e^{-(\delta(i-1))^{1-a}}) M_i, \tag{3.45}$$

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where $X_1 = R^{-\alpha} \sum_{x \in [0,1]} (R\gamma_x)^a (1 - e^{-(R\gamma_x)^{1-a}}) \mathbb{I}_{\{\gamma_x \in [0, \frac{\delta}{R}]\}}$ and M_i is the number of points of $\mathcal{P}$ in the region $[0,1] \times [\frac{\delta}{R}(i-1), \frac{\delta}{R}i]$ (recall paragraph of (3.10) above).

X_1 can be bounded above by $R^{-\alpha} \sum_{x \in [0,1]} (R\gamma_x) \mathbb{I}_{\{\gamma_x \in [0, \frac{\delta}{R}]\}}$, and this has the same distribution as $R^{-\alpha} \sum_{x \in [0, R^\alpha]} \gamma_x \mathbb{I}_{\{\gamma_x \in [0, \delta]\}}$ for every $R > 0$, by the scale invariance of γ. We can use standard large deviation estimates for the latter expression to conclude that X_1 can be ignored in the limits as $R \to \infty$ and then $\delta \to 0$ (here we may use the existence of a positive exponential moment for $\sum_{x \in [0,1]} \gamma_x \mathbb{I}_{\{\gamma_x \in [0, \delta]\}}$ for any δ, a result that follows as an application of Campbell Theorem – see [27]). We concentrate on the right hand sides of (3.44, 3.45).

We start with (3.44). By the exponential Markov inequality, we get, for given $\theta, \xi > 0$,

$$\mathbb{P}\left(R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} X_i^+ \geq R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} \mathbb{E}X_i^+ + \xi \right) \leq \frac{A}{B} \tag{3.46}$$

where $A = \mathbb{E}e^{\theta \sum_{i=1}^{R\delta^{-1}} X_i^+}$ and $B = e^{\theta \sum_{i=1}^{R\delta^{-1}} \mathbb{E}X_i^+ + R^\alpha \xi}$.

Since M_i, $i \geq 2$, are independent Poisson random variables, we obtain

$$\frac{A}{B} = e^{-R^\alpha \xi \theta + \sum_{i=2}^{R\delta^{-1}} (e^{c_i \theta} - 1 - c_i \theta) \mathbb{E}M_i}, \tag{3.47}$$

where $c_i = (\delta i)^a (1 - e^{-(\delta i)^{1-a}})$.

We choose $\theta = R^{-b}$ with $a < b < \alpha < 2b$. Then, using the estimate

$$\mathbb{E}M_i = \int_{\frac{\delta}{R}(i-1)}^{\frac{\delta}{R}i} \frac{\alpha}{x^{1+\alpha}} dx \leq \frac{R^\alpha}{\delta^\alpha (i-1)^{1+\alpha}}, \tag{3.48}$$

we find that the sum in the exponent in (3.47) is bounded above by

$$\sum_{i=2}^{R\delta^{-1}} \frac{R^\alpha}{\delta^\alpha (i-1)^{1+\alpha}} (c_i R^{-b})^2 \leq 2 \frac{R^{\alpha-2b}}{\delta^{\alpha-2a}} \sum_{i=1}^{R\delta^{-1}} i^{-(1+\alpha-2a)} \tag{3.49}$$

Since the sum on the right of (3.49) is bounded by constant times $R^{2a-\alpha} \vee \log R$, and using the above estimates, we find that the exponent in (3.47) is bounded above by

$$-R^{\alpha-b}\xi + \text{ const } R^{-c'}, \tag{3.50}$$

for some constant $c' > 0$. We can then apply Borel-Cantelli and conclude that for a.e. γ, given $\xi > 0$

$$R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} X_i^+ \leq R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} \mathbb{E}X_i^+ + \xi \tag{3.51}$$

for all large enough R.

Conversely, we can conclude that given $\xi > 0$, for a.e. γ and all R large enough

$$R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} X_i^- \geq R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} \mathbb{E}X_i^- - \xi. \tag{3.52}$$

(3.51) and (3.52) then imply that

$$\liminf_{R\to\infty} R^{-\alpha} \sum_{i=1}^{R\delta^{-1}} \mathbb{E}X_i^- \le \liminf_{R\to\infty} W \le \limsup_{R\to\infty} W \le \limsup_{R\to\infty} R^{-\alpha} \sum_{i=1}^{R\delta^{-1}} \mathbb{E}X_i^+. \qquad (3.53)$$

To conclude, it is enough to verify that

$$\liminf_{\delta\to 0} \liminf_{R\to\infty} R^{-\alpha} \sum_{i=1}^{R\delta^{-1}} \mathbb{E}X_i^- = \limsup_{\delta\to 0} \limsup_{R\to\infty} R^{-\alpha} \sum_{i=1}^{R\delta^{-1}} \mathbb{E}X_i^+ \qquad (3.54)$$

is a (positive finite) constant $\hat{c}$.

We begin with the following estimate.

$$\mathbb{E}X_i^+ = (\delta i)^a(1 - e^{-(\delta i)^{1-a}}) \int_{\frac{\delta}{R}(i-1)}^{\frac{\delta}{R}i} \frac{\alpha}{x^{1+\alpha}} dx \le (\delta i)^a(1 - e^{-(\delta i)^{1-a}}) \frac{\delta}{R} \frac{\alpha}{\left(\frac{\delta}{R}(i-1)\right)^{1+\alpha}} \qquad (3.55)$$

Summing up:

$$
\begin{aligned}
R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} \mathbb{E}X_i^+ &\le R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} (\delta i)^a(1 - e^{-(\delta i)^{1-a}}) \frac{\delta}{R} \frac{\alpha}{\left(\frac{\delta}{R}(i-1)\right)^{1+\alpha}} \\
&= \alpha \sum_{i=2}^{R\delta^{-1}} \delta^{a-\alpha} \frac{1 - e^{-(\delta i)^{1-a}}}{i^{1+\alpha-a}} \left(\frac{i}{i-1}\right)^{1+\alpha}
\end{aligned}
\qquad (3.56)
$$

Now as $R \to \infty$, the latter sum converges to a series, which is readily seen to be an approximation to an integral. We find that

$$\limsup_{\delta\to 0} \limsup_{R\to\infty} R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} \mathbb{E}X_i^+ \le \alpha \int_0^\infty \frac{1 - e^{-x^{1-a}}}{x^{1+\alpha-a}} dx. \qquad (3.57)$$

We similarly find the latter expression as a lower bound for

$$\liminf_{\delta\to 0} \liminf_{R\to\infty} R^{-\alpha} \sum_{i=2}^{R\delta^{-1}} \mathbb{E}X_i^-$$

and (3.54) follows, with the right hand side of (3.57) as the constant $\hat{c}$. $\square$

Remark 16 *In order to get an integrated result, rather than the quenched one above, it is enough to replace $\{\gamma_x, x \in [0,1]\}$ by a suitable version. In view of the right hand side of (3.41), we consider $\{R^{-1}\gamma_x, x \in [0, R^\alpha]\}$, and that expression thus becomes*

$$R^{-\alpha} \sum_{x\in[0,R^\alpha]} \gamma_x^a(1 - e^{-\gamma_x^{1-a}}). \qquad (3.58)$$

Since $\sum_{x\in[0,1]} \gamma_x^a(1 - e^{-\gamma_x^{1-a}})$ is integrable, as can be checked by an application of Campbell Theorem, the Law of Large Numbers yields a quenched result for the version, which implies the integrated result for the original γ.

Remark 17 *A few words about the cases where $\alpha \leq a \leq 1$. When $a > \alpha$, we have that Z is a jump process in $\mathbb{N}^*$ (see Remark 6) with $Z(0) = \gamma_x$ with probability proportional to γ_x^a, $x \in [0,1]$. It follows then that $Z^{(\varepsilon)} \to \infty$ identically almost surely as $\varepsilon \to 0$.*

The case $a = \alpha$ demands more delicate analysis. We have that $\hat{\varphi}^{(\varepsilon)}(\lambda)$ (see 3.38), when scaled with a factor of $|\log \varepsilon|^{-1}$ (instead of $\varepsilon^{\hat{\alpha}} = 1$ in this case), converges to a number r independent of $\lambda > 0$ as $n \to \infty$ in probability, and this is the Laplace exponent of a subordinator which equals 0 for an exponentially distributed amount of time of rate r, and then jumps to ∞, where it stays. One may then argue from this that $Z^{(\varepsilon)} \to \infty$ identically as $\varepsilon \to 0$ in probability.

3.3   Aging in the K process

Theorem 3.2 may be viewed as an aging result for Z, since $\hat{Z}$ is nontrivial and self similar with index 1. Corresponding aging results for two-time correlation functions follow.

Below we consider three examples of correlation functions related to aging, and derive scaling limit/aging results for them as a consequence of Theorem 3.2 (as well as of other results derived above). Other correlation functions can be similarly treated.

Example 1   We start with the time correlation function introduced in [1], which is the one that is usually studied in connection with his model. Let

$$\bar{\Pi}(t, s; \gamma) = \mathbb{P}(\text{no jump of } Z \text{ on } [t, t+s]|\gamma) \tag{3.59}$$

(see Remark 18 below).

Let $\Phi \in D$, let $\mathcal{D}(\Phi)$ denote the set of discontinuities of Φ, that is, $\mathcal{D}(\Phi) = \{t \geq 0 : \Phi(t) \neq \Phi(t-)\}$, and consider $F : D \times (0, \infty) \times (0, \infty) \to \{0, 1\}$ such that

$$F(\Phi; t, s) = 1\{[t, t+s] \cap \mathcal{D}(\Phi) = \emptyset\}. \tag{3.60}$$

Then we have that

$$\bar{\Pi}(\varepsilon t, \varepsilon s; \gamma) = \mathbb{E}[F(Z^{(\varepsilon)}; t, s)|\gamma]. \tag{3.61}$$

Let also

$$\hat{\Pi}(t, s) = \mathbb{P}(\text{no jump of } \hat{Z} \text{ on } [t, t+s]). \tag{3.62}$$

Since deterministic single times are almost surely continuity points of $\hat{Z}$, we have that $F(\cdot; t, s)$ is almost surely continuous under the distribution of $\hat{Z}$. We thus conclude from Theorem 3.2 that if $0 \leq a < \alpha$, then for almost every γ

$$\lim_{\varepsilon \to 0} \bar{\Pi}(\varepsilon t, \varepsilon s; \gamma) = \mathbb{E}[F(\hat{Z}; t, s)] = \hat{\Pi}(t, s). \tag{3.63}$$

The aging phenomenon, namely $\hat{\Pi}(\cdot, \cdot)$ being a (nontrivial) function of the ratio of its arguments, then follows from the self similarity with index 1 (and nontriviality) of $\bar{Z}$, but in this case there is an explicit expression for $\hat{\Pi}$, obtained as follows. One readily checks that the right hand side of (3.62) equals $\mathbb{P}([t, t+s] \cap \mathcal{R}(\hat{\Gamma}) = \emptyset)$, where $\mathcal{R}(\Phi)$ is the range of $\Phi \in D$. Since $\hat{\Gamma}$ is an $\hat{\alpha}$-stable subordinator (see Remark 11), an application of the Dynkin and Lamperti arcsine law theorem for that probability yields

$$\hat{\Pi}(t, s) = \frac{\sin(\pi\hat{\alpha})}{\pi} \int_{s/(t+s)}^{1} \theta^{-\hat{\alpha}} (1 - \theta)^{\hat{\alpha}-1} \, d\theta. \tag{3.64}$$

The limit in (3.63) was first obtained in [1] (as the expression in (3.64)) for the case where $a = 0$. The general case $0 \le a \le 1$ was first studied in [4] (see Theorem 3.3 for the case $a < \alpha$, and Theorem 3.4 for the case $a > \alpha$; the particular limit (3.63) is (7.5) in that reference).

In case $a \ge \alpha$, then the discussion in Remark 17 indicates that the limit in (3.63) is identically 1, and that aging is thus interrupted.

For the next examples, we restrict a to $[0, \alpha)$.

Example 2 Let
$$\bar{R}(t, s; \gamma) = \mathbb{P}(Z(t) = Z(t + s) | \gamma). \tag{3.65}$$
Then, the difference between $\bar{R}(\varepsilon t, \varepsilon s; \gamma) = \mathbb{P}(\hat{Z}_t^{(\varepsilon)} = \hat{Z}_{t+s}^{(\varepsilon)} | \gamma)$ and $\bar{\Pi}(\varepsilon t, \varepsilon s; \gamma)$ is given by

$$\mathbb{P}(\hat{Z}_t^{(\varepsilon)} = \hat{Z}_{t+s}^{(\varepsilon)}; \ \hat{Z}_t^{(\varepsilon)} \neq \hat{Z}_{t+r}^{(\varepsilon)} \text{ for some } r \in [0, s] | \gamma). \tag{3.66}$$

Let $\hat{\mathcal{P}}^{(\varepsilon)}$ and $\hat{\mathcal{P}}$ denote the point processes in $(0, \infty) \times (0, \infty)$ associated to $\hat{S}^{(\varepsilon)}$ and $\hat{S}$, respectively, i.e.,

$$\hat{\mathcal{P}}^{(\varepsilon)} = \left\{ \left(t, \hat{S}_t^{(\varepsilon)} - \hat{S}_{t-}^{(\varepsilon)} \right) : t > 0, \ \hat{S}_t^{(\varepsilon)} - \hat{S}_{t-}^{(\varepsilon)} > 0 \right\}, \ \hat{\mathcal{P}} = \left\{ \left(t, \hat{S}_t - \hat{S}_{t-} \right) : t > 0, \ \hat{S}_t - \hat{S}_{t-} > 0 \right\}. \tag{3.67}$$

The convergence in distribution $\hat{S}^{(\varepsilon)} \to \hat{S}$ argued in the proof of Theorem 3.2 implies that

$$\hat{\mathcal{P}}^{(\varepsilon)} \to \hat{\mathcal{P}} \tag{3.68}$$

as $\varepsilon \to 0$ in distribution (in the point process sense; for almost every γ).

Let also $\hat{\Gamma}^{(\varepsilon)}(t) = \varepsilon^{-1} \Gamma(\varepsilon^{\hat{\alpha}} t)$, $t \ge 0$ (see paragraph of (3.12) above). We have that

$$\hat{\Gamma}^{(\varepsilon)} \to \hat{\Gamma} \tag{3.69}$$

in distribution for almost every γ (see (3.34) above). This claim may be argued as follows. Since $(\hat{\Gamma}_t^{(\varepsilon)})$ is a subordinator, an entirely similar reasoning to the one employed in the proof of Theorem 3.2 may also be employed to establish this result. It also follows from a continuity property of $(\hat{\Gamma}_t^{(\varepsilon)})$ as a function of $(\hat{S}_t^{(\varepsilon)})$ and $\mathcal{T}$ similar to the one established in Lemma 2.1, and similarly proven. We leave the details for the interested reader.

For arbitrary $\delta, T > 0$, consider now the event

$$A_{\delta,T,t,s}^{(\varepsilon)} = \{ \hat{Z}_t^{(\varepsilon)} > \delta, \ \hat{Z}_{t+s}^{(\varepsilon)} > \delta, \ \hat{\Gamma}^{(\varepsilon)}(T) > t + s \}, \tag{3.70}$$

and let $B_{\delta,T}^{(\varepsilon)}$ be the event that there exist two points in $\hat{\mathcal{P}}^{(\varepsilon)} \cap \{ (0, 2T) \times (\delta/2, \infty) \}$ with the same second coordinate. Now one readily gets from the above convergence results that

$$\lim_{\varepsilon \to 0} \mathbb{P}(A_{\delta,T,t,s}^{(\varepsilon)} | \gamma) = \mathbb{P}(\hat{Z}_t > \delta, \ \hat{Z}_{t+s} > \delta, \ \hat{\Gamma}(T) > t + s), \tag{3.71}$$

and this can be made arbitrarily close to 1 by choosing δ and T appropriately. We also have that

$$\lim_{\varepsilon \to 0} \mathbb{P}(B_{\delta,T}^{(\varepsilon)} | \gamma) = \mathbb{P}(B_{\delta,T}), \tag{3.72}$$

where $B_{\delta,T}$ is the event corresponding to $B_{\delta,T}^{(\varepsilon)}$ upon replacing $\hat{\mathcal{P}}^{(\varepsilon)}$ by $\hat{\mathcal{P}}$. The latter probability clearly vanishes. Now, since the intersection of the event in the probability in (3.66) and $A_{\delta,T,t,s}^{(\varepsilon)}$ is contained in $B_{\delta,T}^{(\varepsilon)}$, we conclude from the above that

$$\lim_{\varepsilon \to 0} \bar{R}(\varepsilon t, \varepsilon s; \gamma) = \lim_{\varepsilon \to 0} \bar{\Pi}(\varepsilon t, \varepsilon s; \gamma) = \hat{\Pi}(t, s) \tag{3.73}$$

for almost every γ.

Remark 18 *The* aging *correlation functions*

$$\begin{align}
\Pi(t, s; \gamma) &= \mathbb{P}(\textit{no jump of } Y \textit{ on } [t, t+s]|\gamma), \tag{3.74}\\
R(t, s; \gamma) &= \mathbb{P}(Y(t) = Y(t+s)|\gamma) \tag{3.75}
\end{align}$$

are more widely considered in the literature than their barred versions (3.59) and (3.65) above. In the present case there is almost surely no difference, since a.s. $\gamma_x \neq \gamma_y$ provided $x \neq y$ and $\gamma_x > 0$.

The above examples could be done either by considering the clock processes $\hat{\Gamma}^{(\varepsilon)}$ and $\hat{\Gamma}$ on their own, together with (3.69), in the case of Example 1, or, in the case of Example 2, we used, besides Theorem 3.2, convergence results for S and Γ (in the appropriate scale), and in both examples the limit is a correlation function of the limiting clock process $\hat{\Gamma}$. Our last example is natural from the aging point of view, requires Theorem 3.2 alone, and the limit is not a function of $\hat{\Gamma}$ alone.

Example 3 Let

$$Q(t, s; \gamma) = \mathbb{P}\left(\sup_{r \in [0,t]} Z(r) < \sup_{r \in [0,t+s]} Z(r)|\gamma\right). \tag{3.76}$$

This function was suggested in [14] as a "measure of the prospects for novelty in the system". $\hat{Z}$ is almost surely continuous in single deterministic times, so we have that

$$\lim_{\varepsilon \to 0} Q(\varepsilon t, \varepsilon s; \gamma) = \mathbb{P}\left(\sup_{r \in [0,t]} \hat{Z}(r) < \sup_{r \in [0,t+s]} \hat{Z}(r)\right) =: \hat{Q}(t, s), \tag{3.77}$$

since the function $1\{\sup_{r \in [0,t]} \Phi(r) < \sup_{r \in [0,t+s]} \Phi(r)\}$ is continuous in $\Phi \in D$ for almost every Φ under the distribution of $\hat{Z}$. We note that $\hat{Q}(t, s)$ is a function of the ratio t/s only, by the self similarity of $\hat{Z}$, but an explicit expression is not available, as far as we know, as it is for $\hat{\Pi}(t, s)$.

References

[1] Bouchaud, J.-P. (1992) Weak ergodicity breaking and aging in disordered systems, *J. Phys. I France* **2**, 1705–1713

[2] Bovier, A.; Faggionato, A. (2005) Spectral characterisation of ageing: the REM-like trap model in the complete graph, *Ann. Appl. Probab.* **15**, 1997–2037

[3] Fontes L.R.G.; Mathieu P. (2008) K-Processes,Scaling limit and aging for the trap model in the complete graph, *Ann. Probab.* **36**, No. 4, 1322-1358

[4] Gayrard, V. (2010) Aging in reversible dynamics of disordered systems. I. Emergence of the arcsine law in Bouchauds asymmetric trap model on the complete graph, preprint, arXiv:1008.3855v1 [math.PR]

[5] Derrida, B. (1980) Random-energy model: Limit of a Family of Disordered Models, *Phys. Rev. Lett.* **45**, 79-82

[6] Ben Arous, G.; Bovier, A.; Gayrard, V. (2003) Glauber dynamics of the random energy model. I. Metastable motion on the extreme states, *Comm. Math. Phys.* **235**, no. 3, 379–425

[7] Ben Arous, G.; Bovier, A.; Gayrard, V. (2003) Glauber dynamics of the random energy model. II. Aging below the critical temperature, *Comm. Math. Phys.* **236**, no. 1, 1–54

[8] Černý, J. (2009) Another view on aging in the REM, in Spin Glasses: Statics and Dynamics Summer School, Paris 2007. *Progress in Probability* **62**, 85-101, Birkhauser

[9] Fontes, L.R.G.; Lima, P.H.S. (2009) Convergence of Symmetric Trap Models in the Hypercube. In: XVth International Congress on Mathematical Physics, 2006, Rio de Janeiro. *New Trends in Mathematical Physics*, Springer, 285-297

[10] Gayrard, V. (2010) Aging in reversible dynamics of disordered systems. II. Emergence of the arcsine law in the random hopping time dynamics of the REM, preprint, arXiv:1008.3849

[11] Ben Arous, G.; Bovier, A.; Černý, J. (2008) Universality of the REM for Dynamics of Mean-Field Spin Glasses, *Comm. Math. Phys.* **282**, no. 3, 663-695

[12] Bovier, A.; Gayrard, V. (2010) Convergence of clock processes in random environments and ageing in the p-spin SK model, to appear in *Ann. Probab.*

[13] Fontes, L.R.G.; Gava, R.; Gayrard, V. (2011) A scaling limit for the GREM-like trap model, in preparation

[14] Fontes, L.R.G.; Isopi, M.; Newman, C. M. (2002) Random walks with strongly inhomogeneous rates and singular diffusions: convergence, localization and aging in one dimension, *Ann. Probab.* **30**, 579-604

[15] Ben Arous, G.; Černý, J. (2005) Bouchaud's model exhibits two different aging regimes in dimension one, *Ann. Appl. Probab.* **15**, no. 2, 1161–1192.

[16] Ben Arous, G.; Černý, J. (2007) Scaling limit for trap models on $\mathbb{Z}^d$, *Ann. Probab.* **35**, no. 6, 2356–2384

[17] Ben Arous, G.; Černý, J.; Mountford, T. (2006) Aging in two-dimensional Bouchaud's model, *Probab. Theory Related Fields* **134**, 1–43

[18] Fontes L.R.G.; Mathieu P. (2010) On the dynamics of trap models in $\mathbb{Z}^d$, arXiv:1010.5418v1 [math.PR]

[19] Barlow, M.; Černý, J. (2011) Convergence to fractional kinetics for random walks associated with unbounded conductances, *Probab. Theory Related Fields* **149**, 639–673

[20] Černý, J. (2011) On two-dimensional random walk among heavy-tailed conductances, Electronic Journal of Probability **16**, 293–313

[21] Mourrat, J.-C. (2010) Scaling limit of the random walk among random traps on $\mathbb{Z}^d$, *Ann. Inst. H. Poincar Probab. Statist*, to appear

[22] Ben Arous, G.; Černý, J. (2006) Dynamics of trap models, in *Mathematical Statistical Physics*, proceedings of LXXXIII Les Houches summer school, available in arxiv.org/abs/math/0603344

[23] Jara, M.; Landim, C.; Teixeira, A. (2011) Quenched scaling limits of trap models, *Ann. Probab.* **39**, 176-223

[24] Ethier S. N.; Kurtz T. G. (1986), *Markov processes. Characterization and convergence*, Wiley

[25] Whitt, W. (2002), *Stochastic-process limits*, Springer

[26] Jacod J.; Shiryaev A. N. (1987) *Limit theorems for stochastic processes*, Springer

[27] Kingman J. F. C. (1993) *Poisson processes*, Oxford