Dirichlet processes associated to diffusions.

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Abstract: *Let $(W(t), t \geq 0)$ be a reversible diffusion process with values in $I\!R^d$. We give sufficient conditions on the initial law of W and on a function f for the stochastic process $X(t) := f(W(t))$ to be a Dirichlet process.*

Key Words: *Dirichlet processes, Dirichlet forms, Diffusion processes, Martingale inequalities, Stochastic calculus.*

Abbreviated tittle: Dirichlet processes.

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I. Introduction

Dirichlet processes: let $(W(t), t \geq 0)$ be a diffusion process taking its values in a domain of $I\!R^d$, say Ω; let f be some measurable function defined on Ω and define $X(t) := f(W(t))$. We are concerned with the following question: what can one say about the nature of the stochastic process $(X(t), t \geq 0)$?
It follows from Itô's formula that $X(t)$ is a semi-martingale provided that the coefficients in the generator of W are smooth and provided that f is C_2. This is no longer true if f is less regular, say C_1. In order to be able to deal with more general functions f or with general diffusion processes with non smooth coefficients, M. Fukushima [5] introduced the notion of Dirichlet processes.

Definition 1.1 : *Let $S > 0$ and $t_0^n = 0 < t_1^n, \ldots < t_{n+1}^n = S$ be a sequence of partitions of the interval $[0, S]$ such that $\sup_{i=1\ldots n} t_{i+1}^n - t_i^n \to 0$. A stochastic process $(N(t), 0 \leq t \leq S)$, is said to have 0 quadratic variation along the sequence (t_i^n) if*

$$\sum_{i=0}^{n} (N(t_{i+1}^n) - N(t_i^n))^2 \to 0 \tag{1.1}$$

as n tends to $+\infty$ and in probability.
A stochastic process $(A(t), 0 \leq t \leq S)$ is called a Dirichlet process along the sequence (t_i^n) if it can be written as a sum of a square integrable continuous martingale and a process of 0 quadratic variation.

Dirichlet processes therefore generalize semi-martingales. Various properties are known:
- the space of Dirichlet processes can be endowed with a Banach structure (See Bertoin [1]).
- if A is a Dirichlet process and f a C_1 function, then $f(A)$ is also a Dirichlet process (See Bertoin [1] and Föllmer [3]). Besides the process $f(A)$ satisfies some form of Itô's formula (See Föllmer [3]).

In the sequel, S is fixed and $t_0^n = 0 < t_1^n, \ldots < t_{n+1}^n = S$ is a sequence of partitions of the interval $[0, S]$ such that $\sup_{i=1\ldots n} t_{i+1}^n - t_i^n \to 0$. We shall only consider sequences of partitions (t_i^n) satisfying:

$$\sup_n \sum_{i=1}^{n} (t_{i+1}^n - t_i^n) \frac{1}{\sqrt{t_i^n}} < \infty \tag{1.2}$$

When a process is a Dirichlet process along any sequence of partitions satisfying (1.2), we just call it a *Dirichlet process.*

Diffusions: let Ω be a domain of $I\!R^d$. Let $(a(x) = \sigma(x)^*\sigma(x), x \in \Omega)$ be a symmetric matrix valued function defined in Ω. We assume that a is measurable and locally integrable.

Let us denote by $(W(t), t \geq 0)(P_x, x \in \Omega)$ the Markov process generated by the operator $\mathcal{L} := div(a\nabla)$. This is a formal definition. A more precise one is: consider the bilinear form $\mathcal{E}(f,g) := \int_\Omega \sigma(x)\nabla f(x).\sigma(x)\nabla g(x)dx$ defined for C_∞ functions f and g with compact support contained in Ω. We assume that $\mathcal{E}$ is closable in $L_2(\Omega, dx)$. Let $(\mathcal{E}, \mathcal{F})$ be its closure. $(\mathcal{E}, \mathcal{F})$ is then a local Dirichlet form. By definition, W is the Hunt process whose Dirichlet form on $L_2(\Omega, dx)$ is $(\mathcal{E}, \mathcal{F})$.

Let P_t denote the semi-group associated to $(\mathcal{E}, \mathcal{F})$. We use the notation $P_x, x \in \Omega$ to denote the law of $(W(t), t \geq 0)$ when $W(0) = x$, and P_ν for the law of $(W(t), t \geq 0)$ when the law of $W(0)$ is ν. Let ξ denote the death time of W.

Sufficient conditions for the closability of the Dirichlet form $\mathcal{E}$ will be found in [5], chapter 3. In particular if $\Omega = I\!R^d$ and a is bounded from below - i.e. there exists $\lambda > 0$ s.t. $a \geq \lambda Id$ - then $\mathcal{E}$ is closable.

Following [5], we say that a Borel set has 0 capacity if its hitting time by W is a.s. infinite (when the initial law of W is absolutely continuous). We call *smooth* a Radon probability measure that does not charge sets of 0 capacity.

Very little is known about the sample path properties of W in such a general context. We know that, under P_x for quasi almost all x, $(W(t), t \geq 0)$ is continuous, but, unless we require some additional property on a, W will not be a semi-martingale under P_x. Nevertheless, it is proved in [5] that, if the initial law of W is absolutely continuous, then $X(t)$ is a Dirichlet process for all $f \in \mathcal{F}$.

Proposition 1.2 : *Let $f \in \mathcal{F}$, let ν be a smooth probability measure. Assume that, for some $p > 2$, we have $\int_\Omega |\sigma(x)\nabla f(x)|^p dx < \infty$. Then, under P_ν, the process $(X(t) := f(W(t), 0 \leq t \leq S)$ is a Dirichlet process for all $S > 0$.*

This result may be compared with the claims of Rozkosz [8]. Assuming that a is bounded from above and from below, Rozkosz proved that $X(t)$ is a Dirichlet process for any initial law as soon as $f \in W_{1,p}$, for some $p > d \vee 2$. Proposition 1.2 and Rozkosz's Theorems illustrate the interplay between the regularity of the function f and the initial measure that yield the Dirichlet property. Note that we do not need assume that a is bounded to prove Proposition 1.2. We also refer to [4] and the references quoted in [4] for somehow related results when W is a Brownian motion.

As a consequence of the Proposition, we see that the process X has a quadratic variation: under the assumptions of Proposition 1.2, $\sum_{i=0}^n (X(t_{i+1}^n) - X(t_i^n))^2$ converges in probability towards $\int_0^S ds \; |\sigma\nabla f|(W(s))$.

If ν is a smooth probability measure, then, for all $t > 0$, $P_t\nu$ is absolutely continuous. Hence, for any function $f \in \mathcal{F}$ and any $t > 0$, the process $(X(s), t \leq s)$ is a Dirichlet process. Nevertheless the extension to smooth measures is not done in [5] and does not seem to follow from the arguments of [5].

The only difficulty to prove Proposition 1.2 is therefore to control the oscillations of the process W for short times. In [8], this is achieved through Aronson's and Nash's estimates on the semi-group. Here we cannot use such tools. Our proof will rely on weighted norm martingale inequalities.

Energy inequalities: as we just said, the proof of Proposition 1.2 relies on estimates on the fluctuations of the process X for short times. These estimates have their own interest. Let ν be smooth. Then the measure $P_t\nu$ is absolutely continuous w.r.t. the Lebesgue measure on Ω. Let us define $\nu_t := dP_t\nu/dx$. Then $E_\nu[f(W(t))] = \int_\Omega dx\ \nu_t(x)f(x)$. Let $U\nu(x) := \int_0^\infty \nu_t(x)e^{-t}dt$ be the potential of ν, and let $e(\nu) := \int_\Omega \nu(dx)U\nu(x)$ be its energy. Assume that $e(\nu) < \infty$. We then claim that, for all $S > 0$, for all $\eta > 0$, there exists a constant $C = C(e(\nu), \eta, S)$, such that for all $t \in [0, S]$ and $T > 0$, and for all measurable function f, one has:

$$\begin{aligned} &E_\nu[(X(t+T) - X(t))^2; t+T < \xi] \\ &\qquad \le C\left(\frac{T}{\sqrt{t}}\right)^{\frac{\eta}{1+\eta}} \left(\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x)\right)^{\frac{1}{1+\eta}} \end{aligned} \tag{1.3}$$

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II. Proof: martingale inequalities

From Theorem 2.4.4 in [5], we know that smooth measures can be approximated by measures with bounded potentials. Therefore we may, and will, assume that $U\nu \in L_\infty(\Omega)$.

The proof of Proposition 1.2 relies on the following estimate:

$$\begin{aligned} &\sup_n E_\nu[\sum_{i=1}^n (X(t_{i+1}^n) - X(t_i^n))^2; S < \xi] \\ &\qquad \le C(\int_\Omega dx \int_0^S ds\ \nu_s(x)|\sigma(x)\nabla f(x)|^{2(1+\eta)})^{\frac{1}{1+\eta}} \end{aligned} \tag{2.1}$$

(2.1) holds for any $\eta > 0$, any $S > 0$ and a sequence of partitions satisfying (1.2). The constant C depends on η, S, (t_i^n) and $e(\nu)$, but not on f.
Note that the upper bound in (2.1) is finite when η is small enough. Indeed we have assumed that $\int_\Omega dx\ |\sigma\nabla f|^{2(1+\eta)}(x) < \infty$ for some $\eta > 0$. Besides the function $\int_0^S ds\ \nu_s(x) \le e^S U\nu(x)$ is bounded.

We shall deduce (2.1) from the reversibility property, the Lyons-Zheng representation and weighted norm martingale inequalities.

Proof of (2.1): let $P := \int_\Omega dx\ P_x$ be the law of W when started from the Lebesgue measure on Ω. (It may not be a probability measure.)
Let us fix $T > 0$. Let r_T be the time revearsal operator at time T i.e., given a path $(x(t), t \in I\!R_+)$ with $T < \xi$, $r_T(x)$ is defined by $r_T(x)(s) = x(T-s)$ for $s \leq T$, and $r_T(x)(s) = x(0)$ for $s > T$. We then know from [2], Lemma 2.1 that, for every positive $\mathcal{F}_T$-measurable function F,

$$E[For_T; T < \xi] = E[F; T < \xi] \tag{2.2}$$

According to the Lyons-Zheng decomposition (See [6] or [5], chapter 5.7 for the conservative case, and [2], Theorem 2.2 for the general case), we have

$$X(T) - X(0) = \frac{1}{2}(M_T - M_T or_T)$$

on the set $T < \xi$. Here, under P and also under P_x, for quasi almost all x, M is a continuous martingale with $< M >_t = \int_0^t ds\ |\sigma \nabla f|^2(W(s))$.
Therefore, applying the Markov property and then the reversibility property, we have:

$$\begin{aligned} & E_\nu[(X(t+T) - X(t))^2; t+T < \xi] \\ =& E[\nu_t(W(0))(X(T) - X(0))^2; T < \xi] \\ =& \frac{1}{4} E[\nu_t(W(0))(M_T - M_T or_T)^2; T < \xi] \\ \leq& E[\nu_t(W(0))(M_T)^2; T < \xi] + E[\nu_t(W(T))(M_T)^2; T < \xi] \end{aligned} \tag{2.3}$$

Since M is a martigale under P_x, for quasi almost all x, the first term is bounded by:

$$\begin{aligned} & E[\nu_t(W(0))(M_T)^2; T < \xi] \\ \leq& E[\nu_t(W(0))(M_{T\wedge\xi})^2] \\ =& E[\nu_t(W(0)) \int_0^{T\wedge\xi} ds\ |\sigma \nabla f|^2(W(s))] \\ =& \int_\Omega dx \int_0^T ds\ \nu_t(x) P_s(|\sigma \nabla f|^2)(x) \\ =& \int_\Omega dx \int_t^{t+T} ds\ \nu_s(x) |\sigma \nabla f|^2(x) \end{aligned} \tag{2.4}$$

To handle the second term in (2.3), we need some weighted norm inequalities:

Lemma 2.1 (Weighted norm inequalities): *Let $(M_t, t \geq 0)$ be a continuous martingale defined on some probability space $(X, \mathcal{X}, \mathbb{P})$. Let A be a non negative random variable. For all $p > 0$, $\eta > 0$ and $\delta > 0$, there exists a constant $C = C(p, \eta, \delta)$ which is independent of M and A s.t.*

$$\mathbb{E}[(M_T)^{2p} A] \leq C \mathbb{E}[A^{1+\delta}]^{\frac{\eta}{(1+\delta)(1+\eta)}} \mathbb{E}[< M >_T^{p(1+\eta)} A]^{\frac{1}{1+\eta}} \tag{2.5}$$

for any $T > 0$.
We refer to [7] for the proof. Lemma 2.1 is not stated as here in [7] but it follows from the computation of the proof of Théorème B".
Applying the inequality in the Lemma to the second term in (2.3) with $p = 1$, $\delta = 1$ and $A = \nu_t(W(T))1_{T<\xi}$, we get:

$$\begin{aligned}
&E[\nu_t(W(T))(M_T)^2; T < \xi] \\
\leq &C(\eta) E[(\nu_t(W(T)))^2; T < \xi]^{\frac{\eta}{2(1+\eta)}} E[\nu_t(W(T)) < M >_T^{1+\eta}; T < \xi]^{\frac{1}{1+\eta}} \\
= &C(\eta) (\int_\Omega dx\ \nu_t(x)^2)^{\frac{\eta}{2(1+\eta)}} E[\nu_t(W(T)) (\int_0^T ds\ |\sigma \nabla f|^2(W(s)))^{1+\eta}; T < \xi]^{\frac{1}{1+\eta}} \\
\leq &C(\eta) (\int_\Omega dx\ \nu_t(x)^2)^{\frac{\eta}{2(1+\eta)}} E[\nu_t(W(T)) T^\eta \int_0^T ds\ |\sigma \nabla f|^{2(1+\eta)}(W(s)); T < \xi]^{\frac{1}{1+\eta}} \\
&\qquad \text{by Hölder's inequality} \\
= &C(\eta) (\int_\Omega dx\ \nu_t(x)^2)^{\frac{\eta}{2(1+\eta)}} E[\nu_t(W(0)) T^\eta \int_0^T ds\ |\sigma \nabla f|^{2(1+\eta)}(W(s)); T < \xi]^{\frac{1}{1+\eta}} \\
&\qquad \text{by reversibility} \\
\leq &C(\eta) (\int_\Omega dx\ \nu_t(x)^2)^{\frac{\eta}{2(1+\eta)}} E[\nu_t(W(0)) T^\eta \int_0^T ds\ |\sigma \nabla f|^{2(1+\eta)}(W(s))]^{\frac{1}{1+\eta}} \\
= &C(\eta) (\int_\Omega dx\ \nu_t(x)^2)^{\frac{\eta}{2(1+\eta)}} T^{\frac{\eta}{1+\eta}} (\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x) |\sigma \nabla f|^{2(1+\eta)}(x))^{\frac{1}{1+\eta}} \\
&\qquad \text{by definition of } \nu_t \text{ and the semi group property.}
\end{aligned} \tag{2.6}$$

Using the bounds (2.4) and (2.6), we therefore get:

$$\begin{aligned}
&E_\nu[(X(t+T) - X(t))^2; t + T < \xi] \\
\leq &C(\eta) \{ \int_\Omega dx \int_t^{t+T} ds\ \nu_s(x) |\sigma \nabla f|^2(x) \\
+ &(\int_\Omega dx\ \nu_t(x)^2)^{\frac{\eta}{2(1+\eta)}} T^{\frac{\eta}{1+\eta}} (\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x) |\sigma \nabla f|^{2(1+\eta)}(x))^{\frac{1}{1+\eta}} \}
\end{aligned} \tag{2.7}$$

We first note that, due to Hölder's inequality,

$$\begin{aligned}
&\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x)|\sigma\nabla f|^2(x) \\
\leq&(\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x))^{\frac{\eta}{1+\eta}}(\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x))^{\frac{1}{1+\eta}} \\
=&T^{\frac{\eta}{1+\eta}}(\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x))^{\frac{1}{1+\eta}}
\end{aligned} \tag{2.8}$$

Besides, using the inequality $\int_\Omega dx\nu_t(x)^2 \leq \frac{e(\nu)}{2t}$, (see the lemma in the appendix), we also have:

$$\begin{aligned}
&(\int_\Omega dx\ \nu_t(x)^2)^{\frac{\eta}{2(1+\eta)}} T^{\frac{\eta}{1+\eta}}(\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x))^{\frac{1}{1+\eta}} \\
\leq&C(\frac{T}{\sqrt{t}})^{\frac{\eta}{1+\eta}}(\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x))^{\frac{1}{1+\eta}}
\end{aligned} \tag{2.9}$$

, where C is a constant that depends on η, S and $e(\nu)$ only.

From (2.7), (2.8) and (2.9), (1.3) follows, i.e.

$$\begin{aligned}
&E_\nu[(X(t+T)-X(t))^2; t+T<\xi] \\
&\qquad\qquad \leq C\left(\frac{T}{\sqrt{t}}\right)^{\frac{\eta}{1+\eta}} \left(\int_\Omega dx \int_t^{t+T} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x)\right)^{\frac{1}{1+\eta}}
\end{aligned}$$

Thus

$$\begin{aligned}
&E_\nu[\sum_{i=1}^n (X(t_{i+1})-X(t_i))^2; S<\xi] \\
\leq&\sum_{i=1}^n E_\nu[(X(t_{i+1})-X(t_i))^2; t_{i+1}<\xi] \\
\leq&C\sum_{i=1}^n (\frac{t_{i+1}-t_i}{\sqrt{t_i}})^{\frac{\eta}{2(1+\eta)}}(\int_\Omega dx \int_{t_i}^{t_{i+1}} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x))^{\frac{1}{1+\eta}}
\end{aligned} \tag{2.10}$$

Using Hölder's inequality, we bound this quantity by:

$$\begin{aligned}
&\sum_{i=1}^{n}\Big(\frac{t_{i+1}-t_i}{\sqrt{t_i}}\Big)^{\frac{\eta}{2(1+\eta)}}\Big(\int_\Omega dx\int_{t_i}^{t_{i+1}} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x)\Big)^{\frac{1}{1+\eta}}\\
\leq&\Big(\sum_{i=1}^{n}(t_{i+1}-t_i)\frac{1}{\sqrt{t_i}}\Big)^{\frac{\eta}{1+\eta}}\Big(\sum_{i=1}^{n}\int_\Omega dx\int_{t_i}^{t_{i+1}} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x)\Big)^{\frac{1}{1+\eta}} \qquad (2.11)\\
=&\Big(\sum_{i=1}^{n}(t_{i+1}-t_i)\frac{1}{\sqrt{t_i}}\Big)^{\frac{\eta}{1+\eta}}\Big(\int_\Omega dx\int_{0}^{S} ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x)\Big)^{\frac{1}{1+\eta}}
\end{aligned}$$

Therefore, using the assumption (1.2), we get that

$$\begin{aligned}
&\sup E_\nu\Big[\sum_{i=1}^{n}(X(t_{i+1})-X(t_i))^2; S<\xi\Big]\\
\leq& C\Big(\int_\Omega dx\int_0^S ds\ \nu_s(x)|\sigma\nabla f|^{2(1+\eta)}(x)\Big)^{\frac{1}{1+\eta}}
\end{aligned} \qquad (2.1)$$

■

End of the proof of Proposition 1.2: according to [5], we can write $X(t)-X(0)=M_t+N(t)$ where M is a martingale under P_x, for quasi almost all x, and $N(t)$ is a process of 0 quadratic variation under P_μ, for any absolutely continuous probability μ. Therefore all that remains to prove is that N is also of 0 quadratic variation under P_ν.
Let $s<S$.
We first note that $(N(t_1)-N(0))^2\to 0$ as $t_1\to 0$. Indeed the processes X and M are continuous. Therefore N is also continuous.
Also, since $P_s\nu$ is absolutely continuous, then the Markov property implies that

$$\limsup\sum_{i=0}^{n}1(t_i\geq s)(N(t_{i+1})-N(t_i))^2\to 0 \qquad (2.12)$$

under P_ν, in probability.
From (2.1), we deduce that, for any $\varepsilon>0$, we can choose s small enough so that:

$$\limsup\sum_{i=1}^{n}1(t_{i+1}\leq s)E_\nu[(X(t_{i+1})-X(t_i))^2; s<\xi]\leq\frac{\varepsilon}{6} \qquad (2.13)$$

As in the proof (2.4), we have

$$\begin{aligned}
&\limsup\sum_{i=1}^{n}1(t_{i+1}\leq s)E_\nu[(M(t_{i+1})-M(t_i))^2; s<\xi]\\
\leq& E_\nu\Big[\int_0^s du\ |\sigma\nabla f|^2(X_u)\Big]=\int_\Omega dx\int_0^s du\ \nu_u(x)|\sigma\nabla f|^2(x)
\end{aligned} \qquad (2.14)$$

Therefore, by choosing s small enough, we may also assume that this last quantity is bounded by $\varepsilon/6$.
From (2.13) and (2.14), we therefore get that, if s is small enough

$$\limsup \sum_{i=1}^{n} 1(t_{i+1} \leq s) E_\nu[(N(t_{i+1}) - N(t_i))^2; s < \xi] \leq \frac{2\varepsilon}{3} \tag{2.15}$$

We may also assume that

$$P_\nu[s \geq \xi] \leq \frac{\varepsilon}{3} \tag{2.16}$$

Putting together (2.12), (2.15) and (2.16), we deduce that

$$\begin{aligned}
&\limsup_{n\to\infty} P_\nu[\sum_{i=0}^{n} (N(t_{i+1}) - N(t_i))^2 \geq a] \\
&\leq \limsup_{n\to\infty} P_\nu[\sum_{i=0}^{n} 1(t_{i+1} \leq s)(N(t_{i+1}) - N(t_i))^2 \geq a/2] \\
&\leq \limsup_{n\to\infty} \frac{2}{a} E_\nu[\sum_{i=0}^{n} 1(t_{i+1} \leq s)(N(t_{i+1}) - N(t_i))^2; s < \xi] + P_\nu[s \geq \xi]
\end{aligned}$$

$$\leq \frac{2\varepsilon}{a} + \frac{\varepsilon}{3}$$

And since this is true for all $\varepsilon > 0$, we have proved that N has 0 quadratic variation. ■

Appendix: the following inequality is a quite general property of symmetric Markov semi-groups. Let ν be a probability measure with finite energy. Then, for all $t > 0$, we have:

$$\int_\Omega dx\ \nu_t(x)^2 \leq \frac{e(\nu)}{2t} \tag{2.17}$$

Proof: reading (2.17) on a spectral decomposion, one sees that it is sufficient to prove that, for all $\lambda \geq 0$ and for all $t > 0$, we have

$$e^{-2\lambda t} \leq \frac{1}{(1+\lambda)2t} \tag{2.18}$$

But (2.18) is obvious. ■

Example: we describe an application of Proposition 1.2 in the one dimensional situation. Let $\Omega = I\!R$ and let W be the Hunt process associated with a function $(a(x), x \in I\!R)$ such that a and $1/a$ are locally integrable. These two conditions imply that the symmetric form $\mathcal{E}$ is indeed closable (See [5], chapter 3). Besides, we claim that the Dirac mass at the origin is smooth. We may therefore assume that the process W starts from the origin.
In the next proposition we use the following definition: a stochastic process is a *local Dirichlet process* if it can be written as the sum of a local continuous martingale and a process of 0 quadratic variation. We still assume that condition (1.2) holds. Note that our definition of 'process of 0 quadratic variation' is already 'local'.

Proposition 2.2 : *Assume that there exists $\eta > 0$ such that $a \in L_{1+\eta,loc}$, then the coordinate process $(W(t), t \geq 0)$ is a local Dirichlet process under P_0.*
We first check that the Dirac mass at 0 is smooth. It is sufficient to prove that, for some constants C and C', one has:

$$f^2(0) \leq C\mathcal{E}(f,f) + C' \int_{I\!R} dx\ f^2(x)$$

, where $\mathcal{E}(f,f) = \int dx\ a(x)(f'(x))^2$.
But

$$\begin{aligned} f^2(0) =&(f(R) - \int_0^R f'(x)dx)^2 \\ \leq& 2(\int_0^R dx\ f'(x))^2 + 2f^2(R) \\ \leq& 2(\int_0^R \frac{dx}{a(x)})(\int_0^R dx\ a(x)(f'(x))^2) + 2f^2(R) \\ \leq& 2(\int_0^R \frac{dx}{a(x)})\mathcal{E}(f,f) + 2f^2(R) \end{aligned}$$

Since this is true for all $R > 0$, we also have

$$\begin{aligned} f^2(0) \leq& 2(\int_0^R \frac{dx}{a(x)})\mathcal{E}(f,f) + \frac{2}{R}\int_0^R dx\ f^2(x) \\ \leq& 2(\int_0^R \frac{dx}{a(x)})\mathcal{E}(f,f) + \frac{2}{R}\int_{I\!R} dx\ f^2(x) \end{aligned}$$

We have proved that Dirac masses are smooth and therefore it makes sense to choose 0 has initial point for W.
The function $f(x) = x$ belongs to the local domain of the Dirichlet form $\mathcal{E}$ in the broad sense. (See [5], chapter 5.5). Therefore we have

$$W(t) = M_t + N(t)$$

where M is a martingale additive functional locally of finite energy, and N is an additive functional locally of 0 energy. (Cf [5], Theorem 5.5.1).
For $n \in I\!N$, define $\tau_n = \inf\{t \ s.t. \ W(t) = -n \ or \ W(t) = +n\}$. Let f_n be a smooth function with support contained in $[-2n, +2n]$ and such that $f_n(x) = x$ for $x \in [-n, +n]$. There exists a square integrable martingale, M^n, and an additive functional of 0 energy, N_n, such that we have

$$f_n(W(t)) = f_n(W(0)) + M_t^n + N_n(t)$$

Since this decomposition is unique, we must have, under P_0, for $t < \tau_n$, $M_t^n = M_t$ and $N_n(t) = N(t)$. On the other hand, Proposition 1.2 implies that N_n has 0 quadratic variation under P_0. Therefore $\sum_{i=0}^{n}(N(t_{i+1}^n) - N(t_i^n))^2$ converges to 0 in probability on the set $S < \tau_n$. Since this is true for all n, and since τ_n tends to $+\infty$, we have proved that N has 0 quadratic variation. ■

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